Linear Algebra II, spring term 2019
Solutions to Homework 7

1) Eigenvalues of A: $f_A(t) = \begin{vmatrix} 2-t & 1 & 1 \\ 1 & 2-t & -1 \\ 0 & 0 & 3-t \end{vmatrix} = (3-t)\left((2-t)^2-1\right)$

$$= (3-t)(3-t)(1-t) = (3-t)^2(1-t)$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$

Eigenspaces:

$\lambda_1 = 1$: $A - 1\cdot I_3 = \begin{pmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & -1 \\ 0 & 0 & 3-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \mathcal{E}_1(A) = \ker(A - I_3) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$. Set $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

$\lambda_2 = 3$: $A - 3I_3 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_3(A) = \ker(A - 3I_3) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

Set $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Then $\underline{v} = (v_1, v_2, v_3)$ is an eigenbasis of A, and $A = SDS^{-1}$ where

$$S = S^{\underline{e}}_{\underline{v}} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = [A]_{\underline{v}}.$$

Define $u : \mathbb{R} \to \mathbb{R}^3$ by $u(t) = S^{-1}x(t)$ for all $t \in \mathbb{R}$.

Then $u'(t) = Du(t)$, that is, $\begin{cases} u_1'(t) = u_1(t) \\ u_2'(t) = 3u_2(t) \\ u_3'(t) = 3u_3(t) \end{cases}$ for all $t \in \mathbb{R}$.

Hence $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{t} \\ c_2 e^{3t} \\ c_3 e^{3t} \end{pmatrix}$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Now $x(t) = Su(t) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{t} \\ c_2 e^{3t} \\ c_3 e^{3t} \end{pmatrix} = \begin{pmatrix} c_1 e^{t} + (c_2 + c_3) e^{3t} \\ -c_1 e^{t} + c_2 e^{3t} \\ c_3 e^{3t} \end{pmatrix}$

and the initial value condition $x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ gives

$$\begin{cases} c_1 + c_2 + c_3 = 1 \\ -c_1 + c_2 = 1 \\ c_3 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1 \\ c_3 = 1 \end{cases} \Leftrightarrow \frac{1}{2}\begin{cases} c_1 + c_2 = 0 \\ 2c_2 = 1 \\ c_3 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 + c_2 = 0 \\ c_2 = 1/2 \\ c_3 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = -1/2 \\ c_2 = 1/2 \\ c_3 = 1 \end{cases}$$

Hence $\underline{\underline{x(t) = \begin{pmatrix} -\frac{1}{2} e^{t} + \frac{3}{2} e^{3t} \\ \frac{1}{2} e^{t} + \frac{1}{2} e^{3t} \\ e^{3t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^{3t} - e^{t} \\ e^{3t} + e^{t} \\ 2e^{3t} \end{pmatrix} \text{ for all } t \in \mathbb{R}}}$

2) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear map such that $F^2 = id_{\mathbb{R}^n}$.

Then, for all $x, y \in \mathbb{R}^n$: $F(x) \cdot F(y) \underset{\substack{\uparrow \\ F \text{ symmetric}}}{=} x \cdot F(F(y)) = x \cdot F^2(y) \underset{\substack{\uparrow \\ F^2 = id}}{=} x \cdot y$,

and hence $\underline{\underline{F \text{ is orthogonal.}}}$

3) Let f_p be a solution to the equation $f^{(3)}-3f''-10f'=10$ (*)

Then every solution of (*) is of the form $f=f_h+f_p$, where f_h is a solution of the homogeneous equation $f^{(3)}-3f''-10f'=0$ (H)
Conversely, every f of this form is a solution of (*).

By inspection, we see that $f_p(t)=-t$ is a solution of (*).

The characteristic polynomial of (H) is

$$p(x)=x^3-3x^2-10x=x(x^2-3x-10)=x\left(\left(x-\tfrac{3}{2}\right)^2-\tfrac{9}{4}-\tfrac{40}{40}\right)$$
$$=x\left(\left(x-\tfrac{3}{2}\right)^2-\left(\tfrac{7}{2}\right)^2\right)=x(x-5)(x+2)$$

$\Rightarrow f_1(t)=1$, $f_2(t)=e^{5t}$ and $f_3(t)=e^{-2t}$ are solutions of (H).

Since f_1, f_2, f_3 are linearly independent, and the solution space of (H) has dimension 3, it follows that (f_1, f_2, f_3) is a basis of the solution space.
Hence every solution f_h of (H) can be writen as
$f_h=c_1f_1+c_2f_2+c_3f_3$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Therefore, the general solution of (*) is
$\underline{f(t)=c_1+c_2e^{5t}+c_3e^{-2t}-t}$,

and $f'(t)=5c_2e^{5t}-2c_3e^{-2t}-1$.

The condition $f(0)=0$ gives $c_1+c_2+c_3=0$

and $f'(0)=0$ gives $5c_2-2c_3=1$.

$$\begin{cases} c_1+c_2+c_3=0 \\ 5c_2-2c_3=1 \end{cases} \cdot \tfrac{1}{5} \iff \begin{cases} c_1+c_2+c_3=0 \\ c_2-\frac{2}{5}c_3=\frac{1}{5} \end{cases} \cdot(-1)$$

$$\iff \begin{cases} c_1 \quad +\frac{7}{5}c_3=-\frac{1}{5} \\ c_2-\frac{2}{5}c_3=\frac{1}{5} \end{cases}.$$ Set $\mu=c_3$, then

$$\begin{cases} c_1=-\dfrac{1+7\mu}{5} \\ c_2=\dfrac{1+2\mu}{5} \\ c_3=\mu \end{cases}, \quad \mu\in\mathbb{R}.$$

Hence, the solutions of the equation $\circledast$ satisfying $f(0)=f'(0)=0$

are the functions of the form $\underline{\underline{f(t)=-\dfrac{1+7\mu}{5}+\dfrac{1+2\mu}{5}e^{5t}+\mu e^{-2t}-t}}$

where $\mu\in\mathbb{R}$.