Linear Algebra II, spring term 2019
Solutions to Homework 6

1) $(Ae_1)\cdot(Ae_2) = \frac{1}{5}\begin{pmatrix}4\\3\end{pmatrix}\cdot\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{5^2}(4\cdot 3+3\cdot 4) = \frac{24}{25} \neq 0.$

The matrix A is not orthogonal, since the columns of A are not orthonormal.

$$B^TB = \frac{1}{9}\begin{pmatrix}2&1&2\\-2&2&1\\1&2&-2\end{pmatrix}\begin{pmatrix}2&-2&1\\1&2&2\\2&1&-2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}9&0&0\\0&9&0\\0&0&9\end{pmatrix} = I_2$$

$\Rightarrow$ B is invertible and $B^T = B^{-1}$, so B is orthogonal.

2) $$f_A(t) = \det(A-tI_3) = \begin{vmatrix}2-t&1&1\\1&2-t&-1\\0&0&3-t\end{vmatrix} = \begin{vmatrix}2-t&1&1\\3-t&3-t&0\\0&0&3-t\end{vmatrix}$$

$$= (3-t)^2\begin{vmatrix}2-t&1&1\\1&1&0\\0&0&1\end{vmatrix} = (3-t)^2\big((2-t)-1\big) = (1-t)(3-t)^2$$

$\Rightarrow$ The eigenvalues of A are $\lambda_1=1$ and $\lambda_2=3$.

$\mathcal{E}_1(A)$: $$A-I_3 = \begin{pmatrix}1&1&1\\1&1&-1\\0&0&2\end{pmatrix} \sim \begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix} \sim \begin{pmatrix}1&1&0\\0&0&0\\0&0&1\end{pmatrix}$$

$\Rightarrow$ $\mathcal{E}_1(A) = \ker(A-I_3) = \mathrm{span}\left\{\overset{v_1}{\begin{pmatrix}1\\-1\\0\end{pmatrix}}\right\}$, and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ is a basis of $\mathcal{E}_1(F)$.

$\mathcal{E}_3(A)$: $A-3I_3 = \begin{pmatrix}-1&1&1\\1&-1&-1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}0&0&0\\1&-1&-1\\0&0&0\end{pmatrix} \Rightarrow \mathcal{E}_3(A) = \ker(A-3I_3) = \mathrm{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

so $\left(\overset{v_2}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}1\\0\\1\end{pmatrix}}\right)$ is a basis of $\mathcal{E}_3(A)$.

Now $\underline{v} = (v_1, v_2, v_3)$ is an eigenbasis of A, and $[A]_{\underline{v}} = \begin{pmatrix}1&0&0\\0&3&0\\0&0&3\end{pmatrix} =: D$

Set $S = S^{\underline{e}}_{\underline{v}} = \begin{pmatrix}1&1&1\\-1&1&0\\0&0&1\end{pmatrix}$. We have $A = S^{\underline{e}}_{\underline{v}}[A]_{\underline{v}}S^{\underline{v}}_{\underline{e}} = SDS^{-1}$,

and $A^n = (SDS^{-1})^n = SD^nS^{-1}$

Find S^{-1}: $(S|I_3) = \left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\-1&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}0&2&1&1&1&0\\-1&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc}0&2&0&1&1&-1\\1&-1&0&0&-1&0\\0&0&1&0&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}0&1&0&1/2&1/2&-1/2\\1&-1&0&0&-1&0\\0&0&1&0&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}0&1&0&1/2&1/2&-1/2\\1&0&0&1/2&-1/2&-1/2\\0&0&1&0&0&1\end{array}\right)$

$\sim \left(\begin{array}{ccc|c}1&0&0&\\0&1&0&\underbrace{\begin{matrix}1/2&-1/2&-1/2\\1/2&1/2&-1/2\\0&0&1\end{matrix}}_{S^{-1}}\\0&0&1&\end{array}\right)$ $\qquad S^{-1} = S^{\underline{v}}_{\underline{e}} = \frac{1}{2}\begin{pmatrix}1&-1&-1\\1&1&-1\\0&0&2\end{pmatrix}$

$A = SDS^{-1} = \frac{1}{2}\begin{pmatrix}1&1&1\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&3^n&0\\0&0&3^n\end{pmatrix}\begin{pmatrix}1&-1&-1\\1&1&-1\\0&0&2\end{pmatrix} = \underline{\underline{\frac{1}{2}\begin{pmatrix}3^n+1&3^n-1&3^n-1\\3^n-1&3^n+1&-(3^n-1)\\0&0&2\cdot 3^n\end{pmatrix}}}.$

3) Assume that the vectors $F(e_1), \dots, F(e_n)$ are orthonormal.
Then they are linearly independent (by a result from Linear Algebra I) and thus
$0 = F(x) = x_1 F(e_1) + \dots + x_n F(e_n)$ implies $x = 0$.
Hence $\ker F = \{0\}$, so F – and thus A – is invertible.

For $i = 1, \dots, n$, set $v_i = F(e_i)$. Then $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, so

$$A^T A = \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_n \\ \vdots & \ddots & \vdots \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n \end{pmatrix}.$$

As $v_i \cdot v_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$

we have $A^T A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$, so $A^{-1} = A^T$.

4) Assume that $\underline{u}$ is an orthonormal eigenbasis of F.
Then $[F]_{\underline{u}}$ is a diagonal matrix; in particular, $[F]_{\underline{u}}^T = [F]_{\underline{u}}$.
Now, for all $x, y \in \mathbb{R}^2$:

$$F(x) \cdot y = [F(x)]_{\underline{u}} \cdot [y]_{\underline{u}} = ([F]_{\underline{u}} [x]_{\underline{u}})^T [y]_{\underline{u}} = [x]_{\underline{u}} [F]_{\underline{u}}^T [y]_{\underline{u}} = [x]_{\underline{u}} [F]_{\underline{u}} [y]_{\underline{u}} = x \cdot F(y)$$

$\Rightarrow F$ is symmetric.

For the converse implication, assume that F is symmetric.

First, we show that F has an eigenvalue. Let $A = [F] = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be the matrix of F (it is symmetric because F is symmetric).

Then $f_F(t) = \det(F - t\cdot id_{\mathbb{R}^2}) = \det\begin{pmatrix} a-t & b \\ b & c-t \end{pmatrix} = (a-t)(c-t) - b^2$

$$= t^2 - (a+c)t + ac - b^2 = \left(t - \frac{a+c}{2}\right)^2 - \frac{(a+c)^2}{4} + ac - b^2$$

$$= \left(t - \frac{a+c}{2}\right)^2 - \frac{1}{4}\left(a^2 + 2ac + c^2 - 4ac + 4b^2\right) = \left(t - \frac{a+c}{2}\right)^2 - \underbrace{\frac{1}{4}\left((a-c)^2 + 4b^2\right)}_{\geq 0}$$

$\Rightarrow$ The equation $f_F(t) = 0$ has a zero,
so F has an eigenvalue.

Let u be an eigenvector of F with eigenvalue λ, $\|u\| = 1$,
and $v \in \operatorname{span}\{u\}^{\perp}$, $\|v\| = 1$.
Then $F(v)\cdot u = v \cdot F(u) = v\cdot(\lambda u) = \lambda(v\cdot u) = 0 \Rightarrow F(v) \in \operatorname{span}\{u\}^{\perp} = \operatorname{span}\{v\}$
$\Rightarrow F(v) = \mu v$ for some $\mu \in \mathbb{R}$.

Hence (u,v) is an orthonormal basis of $\mathbb{R}^2$, consisting of eigenvectors of F.