Linear Algebra II, spring term 2019
Solutions to Homework 5

1.a) The eigenvalues of F are the zeroes of the characteristic polynomial
$$f_F(t) = \det(F - t\,\mathrm{id}_{\mathbb{R}^3}) = \det(A - tI_3)$$

$$f_F(t) = \begin{vmatrix} -1-t & 0 & 1 \\ -3 & -t & 1 \\ -4 & 0 & 3-t \end{vmatrix} = (-t)\big((-1-t)(3-t) - (-4)\big) = (-t)(t^2 - 2t + 1)$$
$$= -t(t-1)^2$$

The eigenvalues of F are $\underline{\lambda_1 = 0}$ and $\underline{\lambda_2 = 1}$.

Eigenvectors with eigenvalue $\lambda_1 = 0$:

$$\mathcal{E}_0(F) = \ker(F - 0\cdot\mathrm{id}) = \ker F = \ker A$$

$$A = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \mathcal{E}_0(F) = \mathrm{span}\{e_2\}$, so $\underline{e_2 \text{ is a basis of } \mathcal{E}_0(F)}$.

Eigenvectors with eigenvalue $\lambda_2 = 1$: $\mathcal{E}_1(F) = \ker(A - I_3)$

$$A - I_3 = \begin{pmatrix} -2 & 0 & 1 \\ -3 & -1 & 1 \\ -4 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \mathcal{E}_1(F) = \mathrm{span}\left\{\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\}$, so $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is a basis of $\mathcal{E}_1(F)$.

b) Since $\dim \mathcal{E}_0(F) + \dim \mathcal{E}_1(F) = 1 + 1 = 2 < 3 = \dim \mathbb{R}^3$,
$\underline{F \text{ is not diagonalisable}}$.

2.a) The map P satisfies $\begin{cases} P(u) = u & \text{for all } u \in U, \\ P(v) = 0 & \text{for all } v \in U^\perp. \end{cases}$

If $\lambda \neq 0$ is an eigenvalue of P, and $x \in \mathbb{R}^3$ an eigenvector with eigenvalue λ, then $P(\lambda x) = \lambda P(x) = \lambda \cdot (\lambda u)$, so $P(x) \in \mathcal{E}_\lambda(P)$.
But $P(x) \in U \subset \mathcal{E}_1(P)$ for all $x \in \mathbb{R}^3$, hence $P(x) \in \mathcal{E}_\lambda(P) \cap \mathcal{E}_1(P)$.
Thus either $\lambda = 1$ or $P(x) = 0 \Longrightarrow \lambda = 0$.

Note that $U \neq \{0\}$ (since $\begin{pmatrix}1\\0\\1\end{pmatrix} \in U$) $\Longrightarrow$ 1 is an eigenvalue of P.

$U = \left\{ x \in \mathbb{R}^3 \,\middle|\, x \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} = 0 \right\} \Rightarrow \begin{pmatrix}1\\2\\-1\end{pmatrix} \in U^\perp \Rightarrow 0$ is an eigenvalue of P.

So the eigenvalues of P are 0 and 1.

b) Let $u_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}2\\-1\\0\end{pmatrix}$ and $u_3 = \begin{pmatrix}1\\2\\-1\end{pmatrix}$.

Then u_1, u_2 are linearly independent, $u_1, u_2 \in U \Rightarrow \dim U \geq 2$.
But $\dim U^\perp \geq 1$, $\dim U + \dim U^\perp = 3 \Longrightarrow \dim U = 2$ and $\dim U^\perp = 1$.
Hence (u_1, u_2) is a basis of U, and u_3 is a basis of $U^\perp$.
$\Longrightarrow u_1, u_2, u_3$ are linearly independent
$\Longrightarrow (u_1, u_2, u_3)$ is a basis of $\mathbb{R}^3$, consisting of eigenvectors of P.

c) Let $v_1 = \hat{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\1\end{pmatrix} \in U$

$w_2 = u_2 - (u_2 \cdot v_1) v_1 = \begin{pmatrix}2\\-1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$, $\quad v_2 = \hat{w}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix}1\\-1\\-1\end{pmatrix} \in U$

$v_3 = \hat{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix}1\\2\\-1\end{pmatrix} \in U^\perp$

Then $v_1 \cdot v_2 = v_2 \cdot v_3 = v_3 \cdot v_1 = 0$ and $\|v_1\| = \|v_2\| = \|v_3\| = 1$,
so (v_1, v_2, v_3) is an orthonormal basis of $\mathbb{R}^3$, consisting of eigenvectors of P.

3.a) Assume that $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ are distinct eigenvalues of F, and $v_1,\dots,v_n \in V$ corresponding eigenvectors.

Then $\left.\begin{array}{l} v_1,\dots,v_n \text{ are linearly independent} \\ \dim V = n \end{array}\right\} \Rightarrow (v_1,\dots,v_n)$ is a basis of F.

So F is diagonalisable, and $[F]_{(v_1,\dots,v_n)} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

b) For each $i \leq m$, let $\underline{v}^{(i)} = (v_1^i,\dots,v_{l_i}^i)$.

Assume that $\sum_{i=1}^{m}\sum_{j=1}^{l_i} \lambda_j^i v_j^i = 0$ (where $\lambda_j^i \in \mathbb{R}$).

Setting $u_i = \sum_{j=1}^{l_i} \lambda_j^i v_j^i \in \mathcal{E}_{\lambda_i}(F)$, we have $\sum_{i=1}^{m} u_i = 0$.

Since the vectors u_i are eigenvectors with different eigenvalues, this implies that $u_i = 0$ for all i.

Hence, for all i:
$$\sum_{j=1}^{l_i} \lambda_j^i v_j^i = 0 \xRightarrow{v_1^i,\dots,v_{l_i}^i \text{ lin. indep}} \lambda_1^i = \dots = \lambda_{l_i}^i = 0.$$

Hence, $\lambda_j^i = 0$ for all i,j, implying that $\underline{v}^{(1)} \cup \dots \cup \underline{v}^{(m)}$ is lin. independent.

c) If F is diagonalisable then there exists a basis $(u_1,\dots,u_n)$ of V consisting of eigenvectors.

For every $k \in \{1,\dots,m\}$, set $\Lambda_k = \underline{u} \cap \mathcal{E}_{\lambda_k}(F) = \{u_i \in \underline{u} \mid u_i \in \mathcal{E}_{\lambda_k}(F)\}$, and let l_k be the number of elements in Λ_k.

Every basis vector u_i belongs to one of the sets $\Lambda_k \Rightarrow l_1 + \dots + l_m = n$.

Since $\Lambda_k \subset \underline{u}$, the vectors in Λ_k are linearly independent.

As $\Lambda_k \subset \mathcal{E}_{\lambda_k}(F)$, this implies that $l_k \leq \dim \mathcal{E}_{\lambda_k}(F) = \mathrm{gemu}_F(\lambda_k)$

Thus, $n = l_1 + \dots + l_m = \sum_{k=1}^{m} \dim \mathcal{E}_{\lambda_k}(F) = \sum_{k=1}^{m} \mathrm{gemu}_F(\lambda) \leq \sum_{k=1}^{m} \mathrm{almu}_F(\lambda) \leq n$

$$\Longrightarrow \underline{\sum_{k=1}^{m} \dim \mathcal{E}_{\lambda_k}(F) = n}.$$

Conversely, assume that $\sum_{k=1}^{m} \dim \mathcal{E}_{\lambda_k}(F) = n$.

For each $k \in \{1, \dots, m\}$, choose a basis $\underline{v}^{(k)}$ of $\mathcal{E}_{\lambda_k}(F)$.

Then, by (b), the set $\underline{v} = \underline{v}^{(1)} \cup \dots \cup \underline{v}^{(m)}$ is linearly independent, and it contains $\sum_{k=1}^{m} \dim \mathcal{E}_{\lambda_k}(F) = n = \dim V$ vectors.

Hence, $\underline{v}$ is a basis of V consisting of eigenvectors of F, that is, <u>F is diagonalisable</u>.