Linear Algebra II, spring term 2019
Solutions to Homework 4

1) $$\det A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 4 & 5 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 4 & 5 & 6 & 7 \end{vmatrix} = \underline{\underline{0}}$$

$$\det B = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 0 & 4 & 5 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 5 & 6 \end{vmatrix} = \begin{vmatrix} -3 & 0 & -1 & -2 & -3 \\ 3 & 0 & 4 & 5 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 & 6 \end{vmatrix} = \begin{vmatrix} -3 & 0 & -1 & -2 & -3 \\ 3 & 0 & 4 & 5 & 6 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 0 & -1 & -2 & -3 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 11 \end{vmatrix} = - \begin{vmatrix} -3 & 0 & -1 & -2 & -3 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 11 \end{vmatrix} = -(-3)\cdot 1\cdot 3\cdot 1\cdot 11 = \underline{\underline{99}}$$

2) For $i = 2, \dots, n$, let $v_i = \begin{pmatrix} v_{i1} \\ \vdots \\ v_{in} \end{pmatrix}$. Given $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, we have

$$d(x) = \det \begin{pmatrix} | & | & & | \\ x & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \det \begin{pmatrix} - & x & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} = \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix}$$

$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, x_{\sigma(1)} \prod_{i=2}^{n} v_{i,\sigma(i)}$. Similarly, for $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ we get

$$d(y) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, y_{\sigma(1)} \prod_{i=2}^{n} v_{i,\sigma(i)}$$

Moreover, $d(x+y) = \begin{vmatrix} — (x+y) — \\ — v_2 — \\ \vdots \\ — v_n — \end{vmatrix} = \sum_{\sigma \in S_n} \text{sign}(\sigma)(x_{\sigma(1)} + y_{\sigma(1)}) \prod_{i=2}^{n} v_{i,\sigma(i)}$

$$= \sum_{\sigma \in S_n} \text{sign}(\sigma) x_{\sigma(1)} \prod_{i=2}^{n} v_{i,\sigma(i)} + \sum_{\sigma \in S_n} \text{sign}(\sigma) y_{\sigma(1)} \prod_{i=2}^{n} v_{i,\sigma(i)} = d(x) + d(y),$$

verifying the first condition for linearity.

Second, for any $\lambda \in \mathbb{R}$,

$$d(\lambda x) = \begin{vmatrix} — (\lambda x) — \\ — v_2 — \\ \vdots \\ — v_n — \end{vmatrix} = \begin{vmatrix} \lambda x_1 & \cdots & \lambda x_n \\ v_{21} & \cdots & v_{2n} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} \text{sign}(\sigma)(\lambda x_{\sigma(1)}) \prod_{i=2}^{n} v_{i,\sigma(i)}$$

$$= \lambda \cdot \sum_{\sigma \in S_n} \text{sign}(\sigma)\, x_{\sigma(1)} \prod_{i=1}^{n} v_{i,\sigma(i)} = \lambda d(x).$$

Hence, $d : \mathbb{R}^n \to \mathbb{R}$ is a linear map.

b) First, note that $d(x) = 0 \iff$ The matrix $\begin{pmatrix} | & | & & | \\ x & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is not invertible

$\iff$ The vectors $x, v_2, \ldots, v_n$ are linearly dependent

$\iff$ $x \in \text{span}\{v_2, \ldots, v_n\}$.

Hence, $\ker d = \text{span}\{v_2, \ldots, v_n\}$. Since the vectors $v_2, \ldots, v_n$ by assumption are linearly independent, it follows that they form a basis $v = (v_2, \ldots, v_n)$ of $\ker d$.

If $x \notin \text{span}\{v_2, \ldots, v_n\}$ then $d(x) \neq 0$. Thus $\text{im}\, d \subset \mathbb{R}$ is a non-zero subspace, so $\text{im}\, d = \mathbb{R}$ and (for example) the number 1 forms a basis of $\text{im}\, d = \mathbb{R}$.

c) If $v_2, \dots, v_n$ are linearly dependent then the matrix $\begin{pmatrix} | & | & & | \\ x & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is not invertible for any $x \in \mathbb{R}^n$, and hence $d(x) = 0$ for all $x \in \mathbb{R}^n$.
This means that $\ker d = \mathbb{R}^n$ and $\operatorname{im} d = \{0\}$.

Hence, $\underline{e} = (e_1, e_2, \dots, e_n)$ is a basis of $\ker(d)$, and $\emptyset$ is a basis of $\operatorname{im} d$.