Linear Algebra II, spring term 2019
Solutions to Homework 3

1) Non-zero patterns in A: $\begin{pmatrix} 4 & -1 & 1 \\ 5 & 0 & 0 \\ 1 & 5 & 7 \end{pmatrix}$.

The blue pattern has two inversions, the red pattern has one inversion. Hence,

$$\det A = 1\cdot 5\cdot 5 - (-1)\cdot 5\cdot 7 = 25+35 = \underline{60}$$

The matrix B has no non-zero patterns, hence $\underline{\underline{\det B = 0}}$.

$$\det C = 1\cdot(-3)\cdot(-1) + 4\cdot 3\cdot(-1) + 1\cdot 3\cdot 7 - 1\cdot(-3)\cdot(-1) - 4\cdot 3\cdot(-1) - 1\cdot 3\cdot 7$$
$$= 3 - 12 + 21 - 3 + 12 - 21 = \underline{\underline{0}}.$$

The matrix D has two non-zero patterns: $\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 5 \end{pmatrix}$

Red: no inversions. Blue: Five inversions.

So $\det D = 1^3\cdot 5 - (-2)\cdot 3\cdot(-1)\cdot 3 = 5 - 18 = \underline{\underline{-13}}.$

2) If a matrix has two identical rows, or two identical columns, then it determinant is zero.
It follows that $V(2) = V(5) = V(7) = 0$.

On the other hand, $V(t)$ is a polynomial in t of degree 3, hence the equation $V(t)=0$ can have at most three solutions.
Thus, $\underline{\underline{V(t)=0 \iff t \in \{2,5,7\}}}$.

3. a) Induction on $n \geq 1$.
$n=1$: Since V is not finitely generated, it is non-zero, and hence there exists a vector $v_1 \in V$, $v_1 \neq 0$. Clearly v_1 is linearly independent.

$n>1$: Assume that there exist linearly independent vectors $v_1, v_2, \dots, v_{n-1} \in V$ (IH).
Since V is not finitely generated, $\mathrm{span}\{v_1,\dots,v_{n-1}\} \neq V$.
Therefore, there exists a vector $v_n \in V$ s.th. $v_n \notin \mathrm{span}\{v_1,\dots,v_{n-1}\}$.

<u>Claim</u>: $v_1,\dots,v_n$ are linearly independent.
Let $\lambda_1 v_1 + \dots + \lambda_{n-1} v_{n-1} + \lambda_n v_n = 0$ $(\lambda_1,\dots,\lambda_n \in \mathbb{R})$.
Then $\lambda_n v_n = -\lambda_1 v_1 - \lambda_2 v_2 - \dots - \lambda_{n-1} v_{n-1} \in \mathrm{span}\{v_1,\dots,v_{n-1}\}$
$\implies \lambda_n = 0$ (since $v_n \notin \mathrm{span}\{v_1,\dots,v_{n-1}\}$).
$\implies \lambda_1 v_1 + \dots + \lambda_{n-1} v_{n-1} = 0 \implies \lambda_1 = \dots = \lambda_{n-1} = 0$, because $v_1,\dots,v_{n-1}$ are linearly independent.
Hence $v_1,\dots,v_n$ are linearly independent.

By induction, we conclude that, <u>for any $n \geq 1$, there exist linearly independent vectors $v_1,\dots,v_n \in V$.</u>

3.b) Let V be a finitely generated vector space, and $U \subset V$ a subspace.
Then there exist $v_1, \dots, v_m \in V$ such that $\text{span}\{v_1, \dots, v_m\} = V$.
Assume that $u_1, \dots, u_\ell \in U$ are linearly independent.
Then $u_1, \dots, u_\ell \in V$ (as $U \subset V$) and, by a lemma from earier in the course, $\ell \leq m$.
But if U were not finitely generated then, by (a), there would exist lists of linearly independent vectors of any (finite) length. Therefore, U must be finitely generated, QED.

(Bonus problem)

4.a) Let $\mu = \min\{ m \in \{0, \dots, n\} \mid \sigma(a) = a \text{ for all } a > m\}$
Proof by induction on μ.

If $\mu = 0$: Then $\sigma = id_{\{1,\dots,n\}}$, and is an empty product of elements of the form $\tau_{i,j}$.

If $\mu > 0$: Assume that every permutation $\sigma' \in S_n$, satisfying $\sigma'(a) = a$ for all $a > \mu - 1$, can be written on the form $\sigma' = \tau_1 \cdots \tau_r$, where $\tau_1 = \tau_{i_1, j_1}, \dots, \tau_r = \tau_{i_r, j_r}$. (IH)

Since σ is bijective and $\sigma(\mu) \neq \mu$, it follows that $\sigma(\sigma(\mu)) \neq \sigma(\mu)$.
Hence $\sigma(\mu) \leq \mu$. Set $\tau = \tau_{\mu, \sigma(\mu)}$. Then the permutation $\tau\sigma$ satisfies $\tau\sigma(\mu) = \tau(\sigma(\mu)) = \mu$ and, for all $a > \mu$, $\tau\sigma(a) = \tau(a) = a$.
By the induction hypothesis, $\tau\sigma = \tau_1 \cdots \tau_r$ for some $\tau_1 = \tau_{i_1, j_1}, \tau_2 = \tau_{i_2, j_2}, \dots, \tau_r = \tau_{i_r, j_r}$.
As $\tau^2 = id$, we have $\sigma = \tau^2\sigma = \tau\tau_1 \cdots \tau_r$.
By induction, <u>the statement holds for all $\sigma \in S_n$.</u>

4.b) <u>Claim</u>: Let $\sigma \in S_n$, $\tau = \tau_{k,\ell} \in S_n$ for some $k, \ell \in \{1, \dots, n\}$.
Then $\operatorname{sign}(\sigma\tau) = -\operatorname{sign}(\sigma)$.

<u>Proof</u>: See Lemma 1.23 (p.17) of Knapp: Basic Algebra (Birkhäuser, 2006).

The result now follows by induction on m:
Let $\sigma \in S_n$. Then, by (a), there exist $\tau_1, \dots, \tau_m \in S_n$, each of the form $\tau_{k,\ell}$ for some $k, \ell \in \{1, \dots, n\}$, such that $\sigma = \tau_1 \cdots \tau_m$.

If $m = 0$: Then $\sigma = id$ and $\operatorname{sign}(\sigma) = 1 = (-1)^0$

If $m > 0$: Assume that $\operatorname{sign}(\tau_1 \cdots \tau_{m-1}) = (-1)^{m-1}$ (IH).
Then, by the claim above, $\operatorname{sign}(\sigma) = \operatorname{sign}(\tau_1 \cdots \tau_{m-1} \tau_m) = -\operatorname{sign}(\tau_1 \cdots \tau_{m-1})$
$= -(-1)^{m-1} \overset{\text{(by IH)}}{=} (-1)^m$.

By induction, it follows that $\underline{\underline{\operatorname{sign}(\sigma) = (-1)^m \text{ holds for all } m \in \mathbb{N}}}$.