Linear algebra II, spring term 2019
Solutions to Homework 2

1.a) For all $f,g \in \mathcal{F}(\mathbb{R},\mathbb{R})$, $x \in \mathbb{R}$: $(f+g)(x) = f(x)+g(x)$.

Therefore, $$\mathcal{E}(f+g) = \begin{pmatrix} (f+g)(1) \\ \vdots \\ (f+g)(n) \end{pmatrix} = \begin{pmatrix} f(1)+g(1) \\ \vdots \\ f(n)+g(n) \end{pmatrix} = \begin{pmatrix} f(1) \\ \vdots \\ f(n) \end{pmatrix} + \begin{pmatrix} g(1) \\ \vdots \\ g(n) \end{pmatrix} = \mathcal{E}(f)+\mathcal{E}(g).$$

Similarly, for all $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$, $\lambda \in \mathbb{R}$:
$$\mathcal{E}(\lambda f) = \begin{pmatrix} (\lambda f)(1) \\ \vdots \\ (\lambda f)(n) \end{pmatrix} = \begin{pmatrix} \lambda f(1) \\ \vdots \\ \lambda f(n) \end{pmatrix} = \lambda \begin{pmatrix} f(1) \\ \vdots \\ f(n) \end{pmatrix} = \lambda \mathcal{E}(f).$$

Thus, $\mathcal{E}: \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathbb{R}^n$ is a linear map.

b) For $i \in \{0,\dots,n\}$:
$$\mathcal{E}(p_i) = \begin{pmatrix} p_i(1) \\ \vdots \\ p_i(n) \end{pmatrix} = \begin{pmatrix} 1 \\ 2^i \\ 3^i \\ \vdots \\ n^i \end{pmatrix} = \sum_{\ell=1}^{n} \ell^i e_\ell = e_1 + 2^i e_2 + 3^i e_3 + \dots + n^i e_n.$$

Hence, $$[\mathcal{E}]_{\mathfrak{p}}^{\mathfrak{e}} = \begin{pmatrix} | & | & & | \\ [\mathcal{E}(p_0)]_{\mathfrak{e}} & [\mathcal{E}(p_1)]_{\mathfrak{e}} & \cdots & [\mathcal{E}(p_n)]_{\mathfrak{e}} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{pmatrix} \in \mathbb{R}^{n\times(n+1)}$$

(the element at position (i,j) in $[\mathcal{E}]_{\mathfrak{p}}^{\mathfrak{e}}$ is $i^{(j-1)}$)

c) First, notice that $c_{\underline{e}} : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 e_1 + x_2 e_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the identity map on $\mathbb{R}^2$: $c_{\underline{e}} = id_{\mathbb{R}^2}$.

Hence, we get that $[\varepsilon]^{\underline{e}}_{\underline{f}} = [c_{\underline{e}}^{-1} \varepsilon c_{\underline{f}}] = [\varepsilon c_{\underline{f}}]$.

Let $f \in \mathcal{P}_2$, $f(t) = a_0 + a_1 t + a_2 t^2$. Then $f = a_0 p_0 + a_1 p_1 + a_2 p_2$, and $[f]_{\underline{f}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

that is, $c_{\underline{f}}^{-1}(f) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$.

$$\begin{tikzcd} \mathcal{P}_2 \arrow[r, "\varepsilon"] & \mathbb{R}^2 \\ \mathbb{R}^3 \arrow[u, "c_{\underline{f}}"] \arrow[ur, "\varepsilon c_{\underline{f}}"'] & \end{tikzcd}$$

Thus, $\varepsilon(f) = \varepsilon c_{\underline{f}} c_{\underline{f}}^{-1}(f) = [\varepsilon]^{\underline{e}}_{\underline{f}} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$.

So $f \in \ker \varepsilon \iff \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \ker [\varepsilon]^{\underline{e}}_{\underline{f}}$, and $\operatorname{im} \varepsilon = \operatorname{im} [\varepsilon]^{\underline{e}}_{\underline{f}}$.

$$[\varepsilon]^{\underline{e}}_{\underline{f}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} = \operatorname{rref}\left([\varepsilon]^{\underline{e}}_{\underline{f}}\right)$$

$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \ker [\varepsilon]^{\underline{e}}_{\underline{f}} \iff \begin{cases} a_0 - 2a_2 = 0 \\ a_1 + 3a_2 = 0 \end{cases}$. Set $a_2 = s$: $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\ker [\varepsilon]^{\underline{e}}_{\underline{f}} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\} \Rightarrow \underline{\underline{\ker(\varepsilon) = \operatorname{span}\{h\}, \text{ where } h(t) = 2 - 3t + t^2}}$.

$\underline{\underline{\operatorname{im}(\varepsilon) = \operatorname{im}([\varepsilon]^{\underline{e}}_{\underline{f}}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = \mathbb{R}^2}}$

2.a) Let $g,h \in \mathcal{F}(\mathbb{R},\mathbb{R})$.

$\forall t \in \mathbb{R}$: $R_f(g+h)(t) = ((g+h)\circ f)(t) = (g+h)(f(t)) = g(f(t)) + h(f(t))$
$= (g\circ f)(t) + (h \circ f)(t) = (g\circ f + h\circ f)(t) = (R_f(g)+R_f(h))(t)$
$\Longrightarrow \underline{R_f(g+h) = R_f(g) + R_f(h)}$

Let $g \in \mathcal{F}(\mathbb{R},\mathbb{R})$, $\lambda \in \mathbb{R}$.

$\forall t \in \mathbb{R}$: $R_f(\lambda g)(t) = ((\lambda g)\circ f)(t) = (\lambda g)(f(t)) = \lambda(g(f(t))) = \lambda((g\circ f)(t))$
$= (\lambda(g\circ f))(t) = (\lambda R_f(g))(t) \Longrightarrow \underline{R_f(\lambda g) = \lambda R_f(g)}$

Hence, $R_f : \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathcal{F}(\mathbb{R},\mathbb{R})$ is a linear map.

b) Claim: R_f is an isomorphism if and only if f is invertible.
In this case, $R_f^{-1} = R_{f^{-1}}$.

Proof: "$\Leftarrow$": Assume that f is invertible. Then, for all $g \in \mathcal{F}(\mathbb{R},\mathbb{R})$:

$R_{f^{-1}} R_f(g) = R_{f^{-1}}(g\circ f) = g \circ \underbrace{f \circ f^{-1}}_{id_{\mathbb{R}}} = g$, and

$R_f R_{f^{-1}}(g) = R_f(g\circ f^{-1}) = g\circ f^{-1}\circ f = g$, hence $R_f R_{f^{-1}} = R_{f^{-1}} R_f = id_{\mathcal{F}(\mathbb{R},\mathbb{R})}$,
so $\underline{R_f \text{ is invertible and } R_f^{-1} = R_{f^{-1}}}$.

"$\Rightarrow$": Assume that R_f is an isomorphism, $T = R_f^{-1}$, and $k = T(id_{\mathcal{F}(\mathbb{R},\mathbb{R})})$.

First, note that for all $g,h \in \mathcal{F}(\mathbb{R},\mathbb{R})$: $R_f(g\circ h) = g\circ h \circ f = g \circ R_f(h)$.
Now, $T(g) = T(g\circ id) = T(g \circ R_f T(id)) = T R_f(g \circ T(id)) = g \circ k = R_k(g)$,
that is, $T = R_k$ (where $k = T(id)$).
Thus, $id = T R_f(id) = R_k R_f(id) = id \circ f \circ k = f\circ k$, and $id = R_f T(id) = R_f R_k(id) = k \circ f$.
This proves that $\underline{f \text{ is invertible}}$, and $k = f^{-1}$, $\underline{R_f^{-1} = T = R_k = R_{f^{-1}}}$.