Linear algebra II, spring term 2019
Solutions to Homework 1

1) We denote by 0 the zero element in V, and by n the function $X \to V$ defined by $n(x) = 0$ for all $x \in X$.

Let $f, g, h \in \mathcal{F}(X,V)$, $\lambda, \mu \in \mathbb{R}$.

Axioms:

(i) $$\big((f+g)+h\big)(x) \overset{\text{(by def of "+" in } \mathcal{F}(X,V))}{=} (f+g)(x) + h(x) = \big(f(x)+g(x)\big) + h(x) \overset{(V \text{ satisfies (i)})}{=} f(x) + \big(g(x)+h(x)\big)$$
$$= f(x) + (g+h)(x) = \big(f+(g+h)\big)(x) \quad \text{for all } x \in X$$
$$\Longrightarrow \underline{(f+g)+h = f+(g+h)}$$

(ii) $$(f+g)(x) = f(x)+g(x) \overset{(V \text{ satisfies (ii)})}{=} g(x)+f(x) = (g+f)(x) \quad \text{for all } x \in X \Longrightarrow \underline{f+g = g+x}$$

(iii) $$(n+f)(x) = n(x)+f(x) = 0+f(x) \overset{(\forall v \in V:\, 0+v=v)}{=} f(x) \quad \text{for all } x \in X \Longrightarrow \underline{n+f=f}$$

(iv) Define a function $-f \in \mathcal{F}(X,V)$ by $(-f)(x) = -f(x)$ for all $x \in X$.
Then $\big(f+(-f)\big)(x) = f(x) + \big(-f(x)\big) = f(x) - f(x) = 0 \in V$ for all $x \in X$
$$\Longrightarrow \underline{f+(-f) = n}$$

(v) $$\forall x \in X:\ \big(\lambda(f+g)\big)(x) = \lambda\big((f+g)(x)\big) = \lambda\big(f(x)+g(x)\big) \overset{\text{(because } V \text{ satisfies (v))}}{=} \lambda f(x) + \lambda g(x)$$
$$= (\lambda f)(x) + (\lambda g)(x) = (\lambda f + \lambda g)(x) \Longrightarrow \underline{\lambda(f+g) = \lambda f + \lambda g}$$

(vi) $$\forall x \in X:\ \big((\lambda+\mu)f\big)(x) = (\lambda+\mu)\cdot f(x) \overset{(V \text{ satisfies (vi)})}{=} \lambda f(x) + \mu f(x) = (\lambda f)(x) + (\mu f)(x) = (\lambda f + \mu f)(x)$$
$$\Longrightarrow \underline{(\lambda+\mu)f = \lambda f + \mu f}$$

名古屋大学大学院多元数理科学研究科

(vii) $\forall x \in X$: $\big(\lambda(\mu f)\big)(x) = \lambda\big((\mu f)(x)\big) = \lambda\big(\mu \cdot f(x)\big) \overset{\text{(V satisfies (vii))}}{=} (\lambda\mu)\cdot f(x) = \big((\lambda\mu)f\big)(x)$

$\Longrightarrow \underline{\lambda(\mu f) = (\lambda\mu) f}$

(viii) $(1 \cdot f)(x) = 1 \cdot f(x) \overset{\text{(V satisfies (viii))}}{=} f(x)$ for all $x \in X$ $\Longrightarrow \underline{1 \cdot f = f}$

<u>Since the axioms (i)–(viii) are satisfied, the set $\mathcal{F}(X,V)$ is a vector space with the given operations.</u>

2.a) We consider it as known that $\mathcal{P}_3$ is a vector space and thus, in particular, that it is closed under addition and multiplication with scalars. The zero element in $\mathcal{P}_3$ is the function $n : \mathbb{R} \to \mathbb{R}$ given by $n(t) = 0$ for all $t \in \mathbb{R}$.

Check that $\mathcal{U}$ is a subspace of $\mathcal{P}_3$:

(i) $n(1) = 0 \Longrightarrow \underline{n \in \mathcal{U}}$

(ii) Let $f, g \in \mathcal{U}$, so that $f(1) = g(1) = 0$.
Then $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0 \Longrightarrow \underline{f+g \in \mathcal{U}}$

(iii) Let $f \in \mathcal{U}$, $\lambda \in \mathbb{R}$. Then $(\lambda f)(1) = \lambda \cdot f(1) = \lambda \cdot 0 = 0 \Longrightarrow \underline{\lambda f \in \mathcal{U}}$.

<u>Hence, $\mathcal{U} \subset \mathcal{P}_3$ is a subspace.</u>

b) Let $f_1, f_2, f_3 \in \mathcal{P}_3$ be defined by $f_1(t) = t - 1$, $f_2(t) = t^2 - 1$, $f_3(t) = t^3 - 1$.
Then $f_1(1) = f_2(1) = f_3(1) = 0$, so $f_1, f_2, f_3 \in \mathcal{U}$.

· The elements $f_1, f_2, f_3 \in \mathcal{U}$ are linearly independent.
Assume that $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = n$ for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

Then, for all $t \in \mathbb{R}$:

$$0 = n(t) = (\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3)(t) = \lambda_1 f_1(t) + \lambda_2 f_2(t) + \lambda_3 f_3(t)$$
$$= \lambda_1 (t-1) + \lambda_2 (t^2-1) + \lambda_3 (t^3-1) = -(\lambda_1+\lambda_2+\lambda_3) + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3$$

If any one of $\lambda_1, \lambda_2, \lambda_3$ is different from 0, then RHS $\to \pm\infty$ as $t \to \infty$.
As LHS $= 0$, it follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
Hence, f_1, f_2, f_3 are linearly independent.

· The polynomial functions $p_i(t) = t^i$, $i \in \{0,1,2,3\}$ span $\mathcal{P}_3$ (that is, $\mathcal{P}_3 = \operatorname{span}\{p_0, p_1, p_2, p_3\}$), hence $\dim \mathcal{P}_3 \leq 4$.
On the other hand $f_1, f_2, f_3 \in \mathcal{U}$ are linearly independent $\Rightarrow \dim \mathcal{U} \geq 3$.
As, for example, $p_0 \in \mathcal{P}_3$ but $p_0 \notin \mathcal{U}$, we have that $\mathcal{U} \subsetneq \mathcal{P}_3$, thus $3 \leq \dim \mathcal{U} < \dim \mathcal{P}_3 \leq 4 \implies \dim \mathcal{U} = 3$ and $\dim \mathcal{P}_3$.

$$\text{Now, } \left.\begin{array}{r} f_1, f_2, f_3 \in \mathcal{U} \text{ linearly independent} \\ \dim \mathcal{U} = 3 \end{array}\right\} \implies \underline{f} = (f_1, f_2, f_3) \text{ is a basis of } \mathcal{U}.$$

3.a) Let $\underline{g} = (f_1, f_2, f_3, p_0)$ (where $p_0(t) = 1$, $\forall t \in \mathbb{R}$).
Then $p_i(t) = t^i = (t^i - 1) + 1 = f_i(t) + p_0(t)$ for all $t \in \mathbb{R}$ $(i = 1,2,3)$

$\implies p_i \in \operatorname{span} \underline{g} \implies \mathcal{P}_3 = \operatorname{span}\{p_0, p_1, p_2, p_3\} \subset \operatorname{span} \underline{g}$

$\Rightarrow \mathcal{P}_3 = \operatorname{span} \underline{g}$.

Since $\underline{g}$ contains four elements, and $\dim \mathcal{P}_3 = 4$, it follows that $\underline{g}$ is a basis of $\mathcal{P}_3$.

名古屋大学大学院多元数理科学研究科

b) As seen in (3a), we have $p_i = f_i + p_0$ for $i \in \{1,2,3\}$.

In other words: $\underline{\underline{[p_1]_g = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}}$, $\underline{\underline{[p_2]_g = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}}}$, $\underline{\underline{[p_3]_g = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}}}$ and,

clearly, $\underline{\underline{[p_0]_g = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}}$.