Linear Algebra II, spring term 2019
Solutions to Midterm exam 4th June

1.a) Assume that $f, g \in U$, so that $f(-t) = -f(t)$ and $g(-t) = -g(t)$ for all $t \in \mathbb{R}$.
Then, for all $t \in \mathbb{R}$,
$(f+g)(-t) = f(-t) + g(-t) = -f(t) - g(t) = -(f(t)+g(t)) = -(f+g)(t)$
$\Rightarrow$ $f+g \in U$.
Given $f \in U$, $\lambda \in \mathbb{R}$, $t \in \mathbb{R}$, we have
$(\lambda f)(-t) = \lambda \cdot f(-t) = -\lambda f(t) = -(\lambda f)(t) \quad \Rightarrow \lambda f \in U$.
Let $n \in \mathcal{P}_3$ be the zero function: $n(t) = 0$, $\forall t \in \mathbb{R}$.
Then $n(-t) = 0 = -n(t)$ for all $t \in \mathbb{R} \quad \Rightarrow n \in U$
Hence, U is a subspace of $\mathcal{P}_3$.

b) Let $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$.
$f \in U \Leftrightarrow \forall t \in \mathbb{R}: f(-t) = -f(t) \Leftrightarrow \forall t \in \mathbb{R}: f(t) + f(-t) = 0$
We have $f(t) + f(-t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_0 - a_1 t + a_2 t^2 - a_3 t^3$
$= 2(a_0 + a_2 t^2)$
so $f \in U \Leftrightarrow \forall t \in \mathbb{R}: a_0 + a_2 t^2 = 0 \Leftrightarrow a_0 = a_2 = 0.$
$\Leftrightarrow f(t) = a_1 t + a_3 t^3.$
In other words, $U = \operatorname{span}\{p_1, p_3\}$, where $p_i(t) = t^i$.

On the other hand, p_1, p_3 are linearly independent:
Let $\lambda_1, \lambda_3 \in \mathbb{R}$ s.th. $\lambda_1 p_1 + \lambda_3 p_3 = 0$, that is, $\lambda_1 p_1(t) + \lambda_3 p_3(t) = 0$ for all $t \in \mathbb{R}$. Then, $\forall t \in \mathbb{R}: \lambda_1 t + \lambda_3 t^3 = 0$, giving

$$\begin{array}{l} (t=1) \\ (t=2) \end{array} \begin{cases} \lambda_1 + \lambda_3 = 0 \\ 2\lambda_1 + 8\lambda_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_3 = 0 \\ 6\lambda_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_3 = 0 \\ \lambda_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_3 = 0 \end{cases}.$$

So p_1, p_3 are linearly independent. Hence (p_1, p_3) is a basis of U.

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2.a) Let $f, g \in \mathcal{P}_3$. Then

$$T(f+g)(t) = \frac{d}{dt}\Big((t+1)\big(f(t)+g(t)\big)\Big) = \frac{d}{dt}\Big((t+1)f(t) + (t+1)g(t)\Big) = \frac{d}{dt}\big((t+1)f(t)\big) + \frac{d}{dt}\big((t+1)g(t)\big)$$
$$= \big(T(f)+T(g)\big)(t)$$

$$\Longrightarrow \underline{T(f+g) = T(f) + T(g)}$$

Let $f \in \mathcal{P}_3$, $\lambda \in \mathbb{R}$. Then $T(\lambda f)(t) = \frac{d}{dt}\big((t+1)\lambda f(t)\big) = \lambda \frac{d}{dt}\big((t+1)f(t)\big) = \lambda T(f)(t)$

$$\Longrightarrow \underline{T(\lambda f) = \lambda T(f)}.$$

Hence, $\underline{\underline{T \text{ is a linear map.}}}$

.b) $T(p_0)(t) = \frac{d}{dt}\big((t+1)\cdot 1\big) = \frac{d}{dt}(t+1) = 1 = p_0(t)$

$$\Longrightarrow \underline{[T(p_0)]_{\mathcal{P}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}.$$

$T(p_1)(t) = \frac{d}{dt}\big((t+1)t\big) = \frac{d}{dt}(t^2+t) = 2t+1 = p_0(t) + 2p_1(t)$

$$\Longrightarrow \underline{[T(p_1)]_{\mathcal{P}} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}}.$$

$T(p_2)(t) = \frac{d}{dt}\big((t+1)t^2\big) = \frac{d}{dt}(t^3+t^2) = 3t^2+2t = 2p_1(t) + 3p_2(t)$

$$\Longrightarrow \underline{[T(p_2)]_{\mathcal{P}} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}}.$$

$T(p_3)(t) = \frac{d}{dt}\big((t+1)t^3\big) = \frac{d}{dt}(t^4+t^3) = 4t^3+3t^2 = 3p_2(t) + 4p_3(t)$

$$\Longrightarrow \underline{[T(p_3)]_{\mathcal{P}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix}}.$$

Hence, $\underline{\underline{[T]_{\mathcal{P}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}}}.$

3) $\lambda \in \mathbb{R}$ is an eigenvalue of $F \iff (F - \lambda \mathrm{id}_{\mathbb{R}^3})$ is not invertible

$$\iff \det(A - \lambda I_3) = 0.$$

$$\det(A-\lambda I_3) = \begin{vmatrix} (-1-\lambda) & 2 & -2 \\ 0 & (1-\lambda) & -2 \\ 0 & -4 & (3-\lambda) \end{vmatrix} = (-1-\lambda)\Big((1-\lambda)(3-\lambda) - (-4)(-2)\Big)$$

$$= (-1-\lambda)\left(\lambda^2 - 4\lambda + 3 - 8\right) = (-1-\lambda)\left((\lambda-2)^2 - 4 - 5\right) = (-1-\lambda)\left((\lambda-2)^2 - 3^2\right)$$

$$= (-1-\lambda)(\lambda-2-3)(\lambda-2+3) = (-1-\lambda)(\lambda-5)(\lambda+1) = -(\lambda+1)^2(\lambda-5)$$

So $\underline{\underline{\lambda \in \mathbb{R} \text{ is an eigenvalue} \iff \lambda = -1 \text{ or } \lambda = 5.}}$

Eigenvectors with eigenvalue $\lambda = -1$: Let $v \in \mathbb{R}^3 \setminus \{0\}$.
Then v is an eigenvector with eigenvalue -1

$$\iff F(v) = -v \iff v \in \ker(A + I_3)$$

$$A + I_3 = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{pmatrix} \sim \begin{array}{c} \tfrac{1}{2} \\ \\ \\ \end{array}\begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \ker(A + I_3) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

So v is an eigenvector with eigenvalue -1 if and only if

$$\underline{\underline{v \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}}}$$

Eigenvectors with eigenvalue $\lambda = 5$

$v \in \mathbb{R}^3 \setminus \{0\}$ is an eigenvector with eigenvalue $5 \iff F(v) = 5v$

$\iff v \in \ker(A - 5I_3)$

$$A - 5I_3 = \begin{pmatrix} -6 & 2 & -2 \\ 0 & -4 & -2 \\ 0 & -4 & -2 \end{pmatrix} \begin{matrix} \frac{1}{2} \\ -1 \\ \end{matrix} \sim \begin{matrix} -1/6 \\ -1/4 \\ \end{matrix} \begin{pmatrix} -6 & 0 & -3 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \ker(A - 5I_3) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

So $v \in \mathbb{R}^3$ is an eigenvector with eigenvalue 5 if and only if

$$v \in \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\} \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

4\. a) First, let $v \in V$ be an eigenvector of F: $F(v) = \lambda_i v$ for some $i \leq m$.
Note that
$$p(t) = (t-\lambda_1)\cdots(t-\lambda_m) = (t-\lambda_1)\cdots(t-\lambda_{i-1})(t-\lambda_{i+1})\cdots(t-\lambda_m)(t-\lambda_i).$$
Therefore,
$$p(F)v = (F-\lambda_1 id)\cdots(F-\lambda_{i-1} id)(F-\lambda_{i+1} id)\cdots(F-\lambda_m id)(F-\lambda_i id)(v)$$

As $(F-\lambda_i id)(v) = F(v) - \lambda_i v = 0$, we have $p(F)v = 0$.

Now, let $v \in V$ be a linear combination of eigenvectors:
$v = c_1 v_1 + \cdots + c_m v_m$, where $c_1, \ldots, c_m \in \mathbb{R}$ and
$F(v_i) = \lambda_i v_i$ for each $i \leq m$.
Since $p(F)$ is a linear map, we have
$$\underline{\underline{p(F)v}} = p(F)(c_1 v_1 + \cdots + c_m v_m) = c_1 p(F) v_1 + \cdots + c_m p(F) v_m = 0 + \cdots + 0 = \underline{\underline{0}}$$
(since each v_i is an eigenvector and thus $p(F)v_i = 0$).

b) Let $F: \mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto Ax$, where $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Then $\lambda \in \mathbb{R}$ is an eigenvalue $\Leftrightarrow 0 = \det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = \lambda^2$,
that is, 0 is the only eigenvalue.

Hence, $p(t) = t - 0 = t$, and $p(F) = F$

Taking $\underline{\underline{v = e_1}}$, we get $p(F)v = F(e_1) = e_2 \neq 0$.

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