Linear Algebra II, spring term 2019
Solutions to final exam 30th July 2019

1.a) Set $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $Ae_1 = e_2$, and

$$Ae_i \cdot Ae_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases},$$ so A is orthogonal.

b) $Ae_3 = e_3$, so e_3 is an eigenvector of A with eigenvalue 1.

c) Set $B = A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $f_B(\lambda) = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda^2+1)(1-\lambda) \Rightarrow \lambda = 1$ is the only eigenvalue of B.

$\mathcal{E}_1(B) = \ker(B - I_3)$

$$B - I_3 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{array}{c} -1 \\ -2 \\ \end{array} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $(B - I_3)x = 0 \Leftrightarrow x_1 = x_2 = 0$, so $\mathcal{E}_1(B) = \operatorname{span}\{e_3\}$, and e_3 is a basis of $\mathcal{E}_1(B)$. Hence $\operatorname{gemu}_B(1) = 1$.

Since $\sum_{\substack{\lambda:\text{eval} \\ \text{of } B}} \operatorname{gemu}_B(\lambda) = \operatorname{gemu}_B(1) = 1 < 3 = \dim \mathbb{R}^3$, the matrix

B is not diagonalisable.

d) Set $C=\begin{pmatrix}1&1&0\\0&2&0\\0&0&3\end{pmatrix}$. $f_C(\lambda)=\begin{vmatrix}1-\lambda&1&0\\0&2-\lambda&0\\0&0&3-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)(3-\lambda)$

$\Rightarrow$ the eigenvalues of C are 1, 2 and 3.

Since $\dim \mathbb{R}^3=3$ and there are three different eigenvalues, it follows that C is diagonalisable.

Since $C^T=\begin{pmatrix}1&0&0\\1&2&0\\0&0&3\end{pmatrix}\neq\begin{pmatrix}1&1&0\\0&2&0\\0&0&3\end{pmatrix}=C$, C is not symmetric.

2) $f_M(\lambda)=\begin{vmatrix}1-\lambda&1\\3&3-\lambda\end{vmatrix}=(1-\lambda)(3-\lambda)-3=\lambda^2-4\lambda=\lambda(\lambda-4)$

$\Rightarrow$ the eigenvalues of M are 0 and 4.

- $M-0I_3 \overset{(-3)}{=} L_1\begin{pmatrix}1&1\\3&3\end{pmatrix}\sim\begin{pmatrix}1&1\\0&0\end{pmatrix} \Longrightarrow v_1=\begin{pmatrix}-1\\1\end{pmatrix}$ is a basis of $\mathcal{E}_0(M)=\ker M$.

- $M-4I_3 \overset{(1)}{=} L_1\begin{pmatrix}-3&1\\3&-1\end{pmatrix}\sim\begin{pmatrix}-3&1\\0&0\end{pmatrix} \Longrightarrow v_2=\begin{pmatrix}1\\3\end{pmatrix}$ is a basis of $\mathcal{E}_4(M)=\ker(M-4I_3)$

Set $\underline{v}=(v_1,v_2)$ - an eigenbasis of M.
Now, $M=SDS^{-1}$, where $D=\begin{pmatrix}0&0\\0&4\end{pmatrix}$ and $S=S^{\underline{e}}_{\underline{v}}=\begin{pmatrix}-1&1\\1&3\end{pmatrix}$

$x_n=M^nx_0=SD^nS^{-1}x_0 \overset{\text{if } n\geq 1}{=} \begin{pmatrix}-1&1\\1&3\end{pmatrix}\begin{pmatrix}0&0\\0&4^n\end{pmatrix}S^{-1}e_1$

$y=S^{-1}e_1 \Leftrightarrow Sy=e_1$. $(S|e_1) \overset{(1)}{=} L_1\left(\begin{array}{cc|c}-1&1&1\\1&3&0\end{array}\right)\sim_{\substack{-1\\1/4}}\left(\begin{array}{cc|c}-1&1&1\\0&4&1\end{array}\right)\sim\left(\begin{array}{cc|c}1&-1&-1\\0&1&1/4\end{array}\right)\sim\left(\begin{array}{cc|c}1&0&-3/4\\0&1&1/4\end{array}\right)$

$\Rightarrow S^{-1}e_1 = \frac{1}{4}\begin{pmatrix}-3\\1\end{pmatrix}$

$\Rightarrow x_n = \begin{pmatrix}-1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 4^n\end{pmatrix}\cdot\frac{1}{4}\begin{pmatrix}-3\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}0\\4^n\end{pmatrix} = \underline{\underline{4^{n-1}\begin{pmatrix}1\\3\end{pmatrix}}}$ for $n\geq 1$

3) $\lambda\in\mathbb{R}$ is an eigenvalue of $T \Leftrightarrow T(f)=\lambda f$ for some $f\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$

$\Leftrightarrow \exists f\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}) : f''-\lambda f = 0 \Leftrightarrow \ker(S_\lambda)\neq 0$, where $S_\lambda = D^2-\lambda\,\mathrm{id}$.

Since $S_\lambda : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is a linear differential operator of order two, $\dim(\ker S_\lambda) = 2 \Rightarrow \ker S_\lambda \neq 0$

$\Rightarrow$ <u>λ is an eigenvalue of T.</u>

· <u>If $\lambda>0$</u>: The characteristic polynomial of S_λ is

$p_{S_\lambda}(x) = x^2-\lambda = (x-\sqrt{\lambda})(x+\sqrt{\lambda})$

$\Rightarrow$ <u>$(e^{\sqrt{\lambda}t}, e^{-\sqrt{\lambda}t})$ is a basis of $\ker(S_\lambda) = \mathcal{E}_\lambda(T)$</u>

· <u>If $\lambda=0$</u>: $p_{S_\lambda}(x) = x^2 \Rightarrow$ <u>$(1,t)$ is a basis of $\mathcal{E}_0(T)$</u>

· <u>If $\lambda<0$</u>: $p_{S_\lambda}(x) = x^2-\lambda$ is irreducible $\Rightarrow$ <u>$(\cos(\sqrt{-\lambda}t), \sin(\sqrt{-\lambda}t))$ is a basis of $\mathcal{E}_\lambda(T)$.</u>

4) "$\Leftarrow$": Assume that $n \in \mathbb{N}$, and that the elements $f(t), tf(t), \ldots, t^n f(t)$ in $\mathcal{F}(\mathbb{R},\mathbb{R})$ are lin. dependent.
Then there exist $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that
$$\sum_{i=0}^{n} \lambda_i t^i f(t) = 0, \quad \forall t \in \mathbb{R}.$$

Thus, for all $t \in \mathbb{R}$: $f(t) \cdot \sum_{i=0}^{n} \lambda_i t^i = 0$

$$\Rightarrow f(t) = 0 \quad \text{or} \quad \sum_{i=0}^{n} \lambda_i t^i = 0$$

Hence, if $f(t) \neq 0$ then $\sum_{i=0}^{n} \lambda_i t^i = 0$, that is, $\mathrm{supp}(f)$ is contained in the set of zeroes of the polynomial $p(t) = \sum_{i=0}^{n} \lambda_i t^i$.
Since p has at most n zeroes in $\mathbb{R}$, it follows that $\underline{\mathrm{supp}(f) \text{ contains at most } n \text{ elements}}$.

"$\Rightarrow$": Let $\mathrm{supp}(f) = \{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n \in \mathbb{R}$.

Let p be the polynomial given by $p(t) = (x_1 - t)(x_2 - t) \cdots (x_n - t)$

Write $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ for some $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$.

$$\Rightarrow p(t) f(t) = (a_0 + a_1 t + \cdots + a_n t^n) f(t) = a_0 f(t) + a_1 t f(t) + \cdots + a_n t^n f(t).$$

$$\left.\begin{array}{l} \cdot \text{ If } t \in \mathrm{supp}(f) \text{ then } p(t) = 0 \Rightarrow p(t) f(t) = 0 \\ \cdot \text{ If } t \notin \mathrm{supp}(f) \text{ then } f(t) = 0 \Rightarrow p(t) f(t) = 0 \end{array}\right\} \Rightarrow \forall t \in \mathbb{R}: p(t) f(t) = 0$$

$$\Rightarrow \forall t \in \mathbb{R}: a_0 f(t) + a_1 t f(t) + \cdots + a_n t^n f(t) = 0$$

Since $a_n \neq 0$, this means that $\underline{\underline{f(t), tf(t), \ldots, t^n f(t) \text{ are lin. dependent.}}}$

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