Linear algebra II, spring term 2018
Solutions to Homework 4

1.a) The equation $f'+2f=0$ has characteristic polynomial
$p(x)=x+2$. $p(x)=0 \iff x=-2$.
So the equation has general solution $f(t)=ce^{-2t}$, $c\in\mathbb{R}$.

$$f(1)=1 \iff ce^{-2}=1 \iff c=e^{2}.$$

The solution is $\underline{f(t)=e^{2-2t}}$.

b) Characteristic polynomial $p(x)=x^2+6x-9$

$$0=p(x)=x^2+6x-9=(x+3)^2-9-9=(x+3)^2-18=(x+3+3\sqrt{2})(x+3-3\sqrt{2})$$
$\iff x=-3\pm3\sqrt{2}$.
The equation has the solutions $\underline{f(t)=c_1e^{(-3-3\sqrt{2})t}+c_2e^{(-3+3\sqrt{2})t}}$
$\underline{c_1, c_2\in\mathbb{R}}$.

c) Characteristic polynomial $\overset{p(x)}{\overset{\shortparallel}{x^3-6x^2+9x}}=x(x^2-6x+9)=$
$=x\left((x-3)^2-9+9\right)=x(x-3)^2$.

General solution: $f(t)=c_1+c_2e^{3t}+c_3te^{3t}$; $c_1, c_2, c_3\in\mathbb{R}$

$$f'(t)=3c_2e^{3t}+c_3e^{3t}+3c_3te^{3t}\Longrightarrow f'(0)=3c_2+c_3$$

$$f''(t)=9c_2e^{3t}+3c_3e^{3t}+3c_3e^{3t}+9c_3te^{3t}\Rightarrow f''(0)=9c_2+6c_3$$

Boundary conditions:

$$\begin{cases} f(0)=0 \\ f'(0)=0 \\ f''(0)=9 \end{cases} \Longleftrightarrow \begin{cases} c_1+c_2 & =0 \\ 3c_2+c_3 & =0 \\ 9c_2+6c_3 & =9 \end{cases} \Longleftrightarrow \begin{cases} c_1+c_2 & =0 \\ 3c_2+c_3 & =0 \\ 3c_3 & =9 \end{cases}$$

$$\Longleftrightarrow \begin{cases} c_1+c_2 & =0 \\ 3c_2+c_3 & =0 \\ c_3 & =3 \end{cases} \Longleftrightarrow \begin{cases} c_1+c_2 & =0 \\ 3c_2 & =-3 \\ c_3 & =3 \end{cases} \Longleftrightarrow \begin{cases} c_1+c_2 & =0 \\ c_2 & =-1 \\ c_3 & =3 \end{cases}$$

$$\Longleftrightarrow \begin{cases} c_1=1 \\ c_2=-1 \\ c_3=3 \end{cases}$$ The solution is $\underline{f(t)=1-e^{3t}+3te^{3t}}$

d) Assume that $f(t)=a\cos t$ $(a\in\mathbb{R})$ satisfies the equation.

$f'(t)=-a\sin t$, $f''(t)=-a\cos t$.

Then the equation is equivalent to:

$$\begin{aligned} & -a\cos t = 4a\cos t+\cos t && \text{for all } t\in\mathbb{R} \\ \Longleftrightarrow\ & 5a\cos t+\cos t=0 && \forall t \\ \Longleftrightarrow\ & (5a+1)\cos t=0 && \forall t \\ \Longleftrightarrow\ & a=-1/5. \end{aligned}$$

Hence, $\underline{f_p(t)=-\frac{1}{5}\cos t}$ is a solution.

The homogeneous equation $f''-4f=0$ has characteristic polynomial $p(x)=x^2-4=(x-2)(x+2)$, and thus solution

$f_h(t)=c_1e^{2t}+c_2e^{-2t}$; $c_1,c_2\in\mathbb{R}$.

Hence, the general solution to the inhomogeneous eq. is $\underline{\underline{f(t)=c_1e^{2t}+c_2e^{-2t}-\frac{1}{5}\cos t}}$ $c_1,c_2\in\mathbb{R}$.

2\. a) Eq. $f'+2f=0$: $\underline{T=D+2\,id}$

Eq. $f''+6f'-9f=0$: $\underline{T=D^2+6D-9\,id}$

Eq. $f'''-6f''+9f'=0$: $\underline{T=D^3-6D^2+9D}$

Eq. $f''(t)=6f(t)+\cos t \iff f''(t)-6f(t)=\cos t$.
Homogeneous eq: $f''-6f=0$: $\underline{T=D^2-6\,id}$

b) In (1a)-(1c), the equations are homogeneous, and in each case a basis of $\ker T$ can be read off from the general solution (without boundary conditions):

(1a): $\underline{e^{-2t}}$ is a basis of $\ker T$;
(1b): $\underline{(e^{(-3-3\sqrt{2})t}, e^{(-3+3\sqrt{2})t})}$ is a basis of $\ker T$;
(1c): $\underline{\underline{(1, e^{3t}, te^{3t})}}$ is a basis of $\ker T$.

(1d): The homogeneous eq. $f''-4f=0$ has solution $f_h(t)=c_1e^{2t}+c_2e^{-2t}$,

and hence $\underline{\underline{(e^{2t}, e^{-2t})}}$ is a basis of $\ker T$.

3.a) $\begin{cases} a_{n+2} = 2a_{n+1} + 3a_n \\ a_{n+1} = a_{n+1} \end{cases} \Longrightarrow \begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix},$

that is, $\underline{\underline{A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}}}$ works.

b) $\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Try to diagonalise A:

Eigenvalues: $f_A(t) = \det(A - \lambda I_2) = \begin{vmatrix} 2-t & 3 \\ 1 & -t \end{vmatrix} = (2-t)(-t) - 3$

$$= t^2 - 2t - 3 = (t-1)^2 - 1 - 3 = (t-1)^2 - 4 = (t-3)(t+1)$$

$\Longrightarrow$ The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$.

Eigenspaces: $\lambda_1 = -1$: $A - (-1)I_2 = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \mathcal{E}_{-1}(A) = \operatorname{span}\{v_1\}$, where $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\lambda_2 = 3$: $A - 3I_2 = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_3(A) = \operatorname{span}\{v_2\}$, where $v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Now $\underline{v} = (v_1, v_2)$ is an eigenbasis, and setting $S = S_{\underline{v}}^{\underline{e}} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$

we have $S^{-1}AS = S_{\underline{e}}^{\underline{v}} A S_{\underline{v}}^{\underline{e}} = [A]_{\underline{v}} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}}_{=: D}$

and thus $A = SDS^{-1}$

Now $A^n = (SDS^{-1})^n = SD^nS^{-1} \Rightarrow \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = SD^nS^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Compute $S^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\quad \begin{pmatrix} x \\ y \end{pmatrix} = S^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1/4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1/4 \\ 0 & 1 & 1/4 \end{array}\right)$$

$\Rightarrow S^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence $\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = SD^nS^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}^n \left(\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$$= \frac{1}{4}\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n \\ 3^n \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (-1)^n + 3^{n+1} \\ 3^n - (-1)^n \end{pmatrix}$$

$\Rightarrow \underline{a_n = \frac{1}{4}\left(3^n - (-1)^n\right)}$ for all $n \in \mathbb{N}$.

4) Prove by induction that $f_0, \dots, f_{m-1} \in \ker T$:

Base case: $m=1$: $f_0(t) = e^{at} \in \ker(D - a\,\mathrm{id})$ by known results.

Induction hypothesis: Assume that $f_0, \dots, f_{m-2} \in \ker(D - a\,\mathrm{id})^{m-1}$.

As $\ker(D - a\cdot\mathrm{id})^{m-1} \subset \ker(D - a\,\mathrm{id})^m = \ker T$,
it follows that $f_0, \dots, f_{m-2} \in \ker T$.

As for f_{m-1}, we have

$$T(f_{m-1}) = (D - a\,\mathrm{id})^m (f_{m-1}) = (D - a\cdot\mathrm{id})^{m-1} (D - a\,\mathrm{id})(f_{m-1})$$

$$(D - a\cdot\mathrm{id})(f_{m-1})(t) = f_{m-1}'(t) - a f_{m-1}(t) = \frac{d}{dt}\left(t^{m-1} e^{at}\right) - a t^{m-1} e^{at}$$

$$= (m-1) t^{m-2} e^{at} + a t^{m-1} e^{at} - a t^{m-1} e^{at} = (m-1) t^{m-2} e^{at} = (m-1) f_{m-2}(t).$$

So $(D - a\cdot\mathrm{id})(f_{m-1}) = (m-1) f_{m-2} \in \ker(D - a\cdot\mathrm{id})^{m-1}$ by the induction hypothesis.

$$\Rightarrow T(f_{m-1}) = (D - a\cdot\mathrm{id})^{m-1} (D - a\cdot\mathrm{id})(f_{m-1}) = 0$$

that is, $f_{m-1} \in \ker T$.

By induction, we have shown that $f_0, \dots, f_{m-1} \in \ker T$ holds for all $m \geq 1$.

Next, to show that $f_0, \dots, f_{m-1}$ are linearly independent, assume that $\lambda_0 f_0 + \dots + \lambda_{m-1} f_{m-1} = 0$ for some $\lambda_0, \dots, \lambda_{m-1} \in \mathbb{R}$.

$\forall t \in \mathbb{R}$:

$$0 = \sum_{i=0}^{m-1} \lambda_i f_i(t) = \sum_{i=0}^{m-1} \lambda_i t^i e^{at} = e^{at}\left(\sum_{i=0}^{m-1} \lambda_i t^i\right)$$

$$\Longleftrightarrow \sum_{i=0}^{m-1} \lambda_i t^i = 0 \quad \text{for all } t \in \mathbb{R}$$

$$\Longleftrightarrow \lambda_i = 0 \quad \text{for all } i = 0, 1, \dots, m-1,$$ since the elements $1, t, t^2, \dots, t^{m-1}$ are lin. indep.

Hence $\underline{f_0, f_1, \dots, f_{m-1}} \in \ker T$ $\underline{\text{are linearly independent}}$.

From the lectures we know that $\dim(\ker T) = m$.

<u>Hence, $(f_0, f_1, \dots, f_{m-1})$ is a basis of $\ker T$.</u>