Linear algebra II, spring term 2018
Solutions to Homework 3

1.a) $Av = \begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = 2v \Rightarrow v$ is an eigenvector of A, with eigenvalue 2.

b) The characteristic polynomial of A is

$$f_A(t) = \det(A - \lambda I_3) = \begin{vmatrix} 4-t & 1 & 1 \\ -5 & -t & -3 \\ -1 & -1 & 2-t \end{vmatrix} = \begin{vmatrix} 3-t & 0 & 3-t \\ -5 & -t & -3 \\ -1 & -1 & 2-t \end{vmatrix}$$

$$= (3-t)\cdot\begin{vmatrix} 1 & 0 & 1 \\ -5 & -t & -3 \\ -1 & -1 & 2-t \end{vmatrix} = (3-t)\Big((-t)(2-t) - (-3)(-1) + (-5)(-1) - (-1)(-t)\Big)$$

$$= (3-t)\left(t^2 - 2t - 3 + 5 - t\right) = (3-t)\left(t^2 - 3t + 2\right)$$

$$= (3-t)\left(\left(t - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{8}{4}\right) = (3-t)\left(\left(t - \tfrac{3}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right) = (3-t)(t-2)(t-1)$$

The eigenvalues of A are the zeroes of $f_A(t)$, that is, the numbers 1, 2 and 3.

<u>Eigenvectors corresponding to the eigenvalue 1:</u>

$$A - 1\cdot I_3 = \begin{pmatrix} 3 & 1 & 1 \\ -5 & -1 & -3 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 4 \\ 0 & 4 & -8 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 4 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_1(A) = \ker(A - 1\cdot I_3) = \operatorname{span}\Big\{\underbrace{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}_{u_1}\Big\}$$

Eigenvectors corresponding to the eigenvalue 2:

$$A-2I_3=\begin{pmatrix}2&1&1\\-5&-2&-3\\-1&-1&0\end{pmatrix}\sim\begin{pmatrix}0&-1&1\\0&3&-3\\-1&-1&0\end{pmatrix}\sim\begin{pmatrix}0&-1&1\\0&0&0\\1&1&0\end{pmatrix}$$

$$\sim\begin{pmatrix}0&-1&1\\0&0&0\\1&0&1\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&1&-1\\0&0&0\end{pmatrix}\Rightarrow \ker(A-2I_3)=\operatorname{span}\left\{\underbrace{\begin{pmatrix}-1\\1\\1\end{pmatrix}}_{u_2}\right\}$$

Eigenvalues corresponding to the eigenvalue 3:

$$A-3I_3=\begin{pmatrix}1&1&1\\-5&-3&-3\\-1&-1&-1\end{pmatrix}\sim\begin{pmatrix}1&1&1\\0&2&2\\0&0&0\end{pmatrix}\sim\begin{pmatrix}1&1&1\\0&1&1\\0&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}$$

$$\Rightarrow \mathcal{E}_3(A)=\ker(A-3I_3)=\operatorname{span}\left\{\underbrace{\begin{pmatrix}0\\-1\\1\end{pmatrix}}_{u_3}\right\}$$

$u_1=\begin{pmatrix}-1\\2\\1\end{pmatrix}$ is a basis of $\mathcal{E}_1(A)$

$u_2=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ is a basis of $\mathcal{E}_2(A)$

$u_3=\begin{pmatrix}0\\-1\\1\end{pmatrix}$ is a basis of $\mathcal{E}_3(A)$

The eigenvalues of A are 1, 2, and 3

c) $\underline{u}=(u_1,u_2,u_3)$ is an eigenbasis of A, hence A is diagonalisable.

For each eigenvalue $\lambda\in\{1,2,3\}$, $\mathrm{almu}_A(\lambda)=\mathrm{gemu}_A(\lambda)=1$

2) $\underline{R}$: If $v \in \mathbb{R}^2$ is a non-zero vector then v, Rv are non-zero and orthogonal, and hence linearly independent.
It follows that v is not an eigenvector of R;
i.e., R has no eigenvalues and no eigenvectors

$\underline{P}$: $Pu = u$, so u is an eigenvector with eigenvalue 1.
The vector $w = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$ is non-zero (since $u \neq 0$), and $w \cdot u = 0$
$\Rightarrow Pw = 0$ Hence w is an eigenvector with eigenvalue 0.
Since $\dim \mathbb{R}^2 = 2$, there can exist no more than 2 linearly independent eigenvectors. Hence, 0 and 1 are the only eigenvalues, and $\begin{cases} \mathcal{E}_0(P) = \mathrm{span}\{w\} \\ \mathcal{E}_1(P) = \mathrm{span}\{u\} \end{cases}$

$\underline{R^2}$: As $R^2 = -id_{\mathbb{R}^2}$, every non-zero vector is an eigenvector with eigenvalue -1.

$\underline{P^2}$: $P^2 = P$, because $\begin{cases} P^2u = Pu \\ P^2w = 0 = Pw \end{cases}$ and (u,w) is a basis of $\mathbb{R}^2$.

Hence, as with P, the eigenvalues are 0 and 1, and $\mathcal{E}_0(P^2) = \mathrm{span}\{w\}$, $\mathcal{E}_1(P^2) = \mathrm{span}\{u\}$.

$\underline{R^{100}}$: $R^{100} = (R^2)^{50} = (-id_{\mathbb{R}^2})^{50} = (-1)^{50} id_{\mathbb{R}^2} = id_{\mathbb{R}^2}$. Hence, every non-zero vector is an eigenvector with eigenvalue 1.

$\underline{PR}$: $PR(u) = P(w) = 0$,
$PR(w) = P(-u) = -u$.

The vectors u and w form a basis of $\mathbb{R}^2$, and $[PR]_{(u,w)} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

Characteristic polynomial:

$$f_{PR}(t) = \det(PR - t\,id_{\mathbb{R}^2}) = \det([PR]_{(u,w)} - tI_2) = \begin{vmatrix} -t & -1 \\ 0 & -t \end{vmatrix} = t^2$$

$\Longrightarrow$ $\underline{0 \text{ is the only eigenvalue of } PR.}$

As $PR \neq 0$, we have $\left.\begin{matrix} \ker(PR) \neq \mathbb{R}^2 \\ u \in \ker(PR) \end{matrix}\right\} \Rightarrow \dim(\ker PR) = 1$, and
$\underline{\mathcal{E}_0(PR) = \ker PR = \operatorname{span}\{u\}}.$

3.a) Assume that $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is an eigenvector of D with eigenvalue $\lambda \in \mathbb{R}$; that is, $f \neq 0$ and $\underbrace{D(f)}_{f'} = \lambda f$.

Setting $g(t) = f(t)e^{-\lambda t}$, we have

$$g'(t) = f'(t)e^{-\lambda t} + f(t)D(e^{-\lambda t}) = \lambda f(t)e^{-\lambda t} + f(t)(-\lambda e^{-\lambda t}) = 0, \quad \forall t$$

$\Rightarrow$ $\underline{g(t) = c \text{ for some } c \in \mathbb{R}}$.

b) From (a) follows that if f is an eigenvector of D with eigenvalue λ, then $f(t)e^{-\lambda t} = c$, i.e., $f(t) = ce^{\lambda t}$, for some $c \in \mathbb{R}$.

Conversely, if $f(t) = ce^{\lambda t}$ $(c \neq 0)$ then $D(f)(t) = c\lambda e^{\lambda t} = \lambda f(t)$ $\forall t$
$\Rightarrow$ $D(f) = \lambda f$.

Hence, every $\lambda \in \mathbb{R}$ is an eigenvalue of D, an the corresponding <u>eigenvectors are the elements of the form $f(t) = ce^{\lambda t}$, for $c \neq 0$.</u>

4.a) Let $f, g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Then

$$\mathcal{J}(f+g)(x) = \int_0^x (f+g)(t)\,dt = \int_0^x (f(t)+g(t))\,dt = \int_0^x f(t)dt + \int_0^x g(t)dt = \mathcal{J}(f)(x) + \mathcal{J}(g)(x)$$

$$\Longrightarrow \mathcal{J}(f+g) = \mathcal{J}(f) + \mathcal{J}(g)$$

For $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ and $\lambda \in \mathbb{R}$:

$$\mathcal{J}(\lambda f)(x) = \int_0^x \lambda f(t)dt = \lambda \int_0^x f(t)dt = \lambda \mathcal{J}(f)(x) \Longrightarrow \mathcal{J}(\lambda f) = \lambda \mathcal{J}(f).$$

Hence, $\mathcal{J}$ is a linear map.

b) For any $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $D\mathcal{J}(f)(x) = \frac{d}{dx}\int_0^x f(t)dt = f(x)$ by the fundamental theorem of calculus.

I.e., $D\mathcal{J}(f) = f$ and $D\mathcal{J} = \mathrm{id}_{\mathcal{C}^\infty(\mathbb{R},\mathbb{R})}$.

On the other hand, if f is a non-zero constant function ($f(t)=c$, $\forall t$, $c \neq 0$), then $D(f)=0$ and hence

$$\mathcal{J}D(f) = \mathcal{J}(0) = 0 \neq f$$

(since $\mathcal{J}$ is a linear map)

$\Longrightarrow \mathcal{J}D \neq \mathrm{id}_{\mathcal{C}^\infty(\mathbb{R},\mathbb{R})}$

c) First, notice that 0 is not an eigenvalue of $\mathcal{J}$:

if $\mathcal{J}(f) = 0$ then $\forall x \in \mathbb{R}: 0 = \mathcal{J}(f)(x) = \int_0^x f(t)dt = F(x) - F(0)$
for some antiderivative F of f.

$\Longrightarrow F(x) = F(0)$ for all $x \in \mathbb{R}$, so F is a constant function
and $f(x) = F'(x) = 0$ for all $x \in \mathbb{R}$, that is, $f = 0$.

Hence $\mathcal{J}(f) \neq 0$ whenever $f \neq 0$, so 0 is not an eigenvalue.

Second, assume that $\lambda \neq 0$ is an eigenvalue of $\mathcal{J}$, and f a corresponding eigenvector.

Then $f = id_{\mathcal{C}^\infty(\mathbb{R},\mathbb{R})}(f) \underset{\uparrow}{=} D\mathcal{J}(f) = D(\lambda f) = \lambda D(f)$
(by (b))

$\Longrightarrow D(f) = \frac{1}{\lambda} f$

$\Rightarrow f$ is an eigenvector of D with eigenvalue $\frac{1}{\lambda}$. $\Longrightarrow \underline{f(t) = ce^{\lambda^{-1}t}}$.
(*)

On the other hand, $\lambda f(x) = \mathcal{J}(f)(x) = \int_0^x f(t)dt$

$\Longrightarrow \lambda f(0) = \int_0^0 f(t)dt = 0 \overset{(\lambda \neq 0)}{\Longrightarrow} \underline{f(0) = 0}$.
(**)

By (*) and (**), $0 = f(0) = ce^{\lambda^{-1}\cdot 0} = ce^0 = c \Rightarrow f = 0$.

But this contradicts that f is an eigenvector, and hence,

<u>the linear map $\mathcal{J}$ does not have any eigenvalues.</u>