Linear algebra II, spring term 2018
Solutions to Homework 2

1) Let $u_1 = Ae_1 = \begin{pmatrix}1\\2\\2\\0\end{pmatrix}$, $u_2 = Ae_2 = \begin{pmatrix}2\\1\\-2\\0\end{pmatrix}$, $u_3 = Ae_3 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$.

Perform Gram-Schmidt on the vectors u_1, u_2, u_3:

$$v_1 = \hat{u}_1 = \frac{1}{3}u_1 = \frac{1}{3}\begin{pmatrix}1\\2\\2\\0\end{pmatrix} \qquad \text{Hence,} \qquad u_1 = 3v_1$$

$$w_2 = u_2 - (u_2 \cdot v_1)v_1, \qquad u_2 \cdot v_1 = \frac{1}{3}\left(\begin{pmatrix}1\\2\\2\\0\end{pmatrix} \cdot \begin{pmatrix}2\\1\\-2\\0\end{pmatrix}\right) = 0$$

$$w_2 = u_2, \quad v_2 = \hat{w}_2 = \frac{1}{3}u_2 = \frac{1}{3}\begin{pmatrix}2\\1\\-2\\0\end{pmatrix} \longmapsto u_2 = 3v_2.$$

$$w_3 = u_3 - (u_3 \cdot v_1)v_1 - (u_3 \cdot v_2)v_2$$

$$u_3 \cdot v_1 = \frac{1}{3}(u_3 \cdot u_1) = \frac{1}{3} \cdot 3 = 1; \qquad u_3 \cdot v_2 = \frac{1}{3}(u_3 \cdot u_2) = 1$$

$$w_3 = u_3 - v_1 - v_2 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\2\\2\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\1\\-2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$v_3 = \hat{w}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \rightsquigarrow u_3 = v_1 + v_2 + w_3 = v_1 + v_2 + v_3$$

Hence, $\underline{v} = (v_1, v_2, v_3) = \left(\frac{1}{3}\begin{pmatrix}1\\2\\2\\0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\1\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$ is the Gram-Schmidt basis corresponding to $\underline{u} = (u_1, u_2, u_3)$, and $S_{\underline{u}}^{\underline{v}} = \begin{pmatrix} | & | & | \\ [u_1]_{\underline{v}} & [u_2]_{\underline{v}} & [u_3]_{\underline{v}} \\ | & | & | \end{pmatrix}$

$$= \begin{pmatrix}3 & 0 & 1\\0 & 3 & 1\\0 & 0 & 1\end{pmatrix}$$

Hence, the QR factorisation of A is

$$A = \underbrace{\frac{1}{3}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{R}.$$

Moreover, $\underline{v} = (v_1, v_2, v_3)$ is an orthonormal basis of $\operatorname{span}\{u_1, u_2, u_3\} = \operatorname{im} A$.

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2) Let $\varepsilon : \mathcal{P}_1 \to \mathbb{R}^3$ be the linear map defined by $\varepsilon(f) = \begin{pmatrix} f(-1) \\ f(0) \\ f(1) \end{pmatrix}$.

Then $\varepsilon(f) = \begin{pmatrix} a-b \\ a \\ a+b \end{pmatrix}$ for $f(t) = a + bt \in \mathcal{P}_1$.

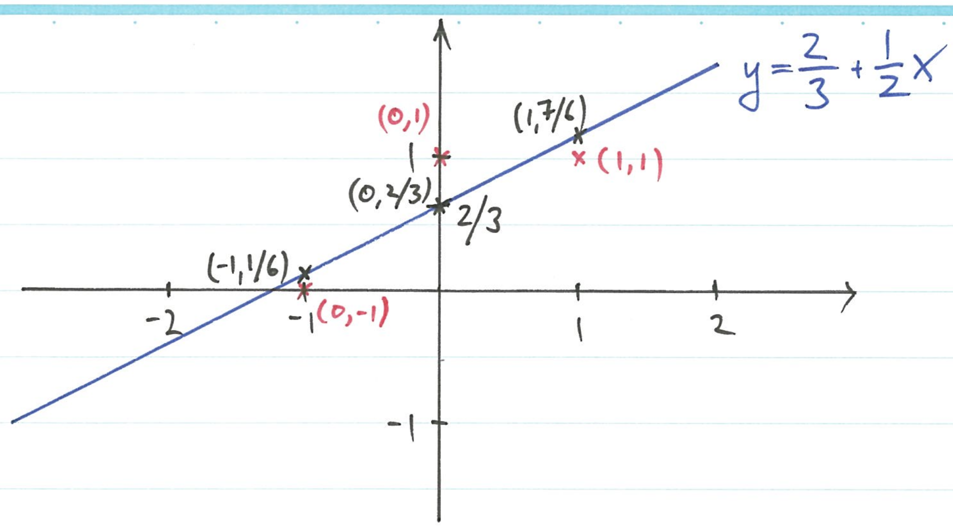
Let $\underline{x} = (1, t)$ – a basis of $\mathcal{P}_1$. Then $[\varepsilon]^{e}_{\underline{x}} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Setting $A = [\varepsilon]^{e}_{\underline{x}}$, the least squares solution is given by $f \in \mathcal{P}_1$ such that $x = [f]_{\underline{x}}$ solves the equation $A^T A = A^T b$, where $b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{matrix} (1/3) \\ (1/2) \end{matrix} \left(\begin{array}{cc|c} 3 & 0 & 2 \\ 0 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/2 \end{array}\right) \qquad x = \begin{pmatrix} 2/3 \\ 1/2 \end{pmatrix}.$$

$\Longrightarrow$ $f(t) = \frac{2}{3} + \frac{1}{2} t$ is the least squares fit of a linear polynomial to the points $(-1, 0)$, $(0, 1)$, $(1, 1)$.



3.a) $x \in U^{\perp} \iff \begin{cases} x \cdot u_1 = 0 \\ x \cdot u_2 = 0 \end{cases} \iff \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} x = 0$

$$\textcircled{-1}\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} \quad \begin{cases} -x_1 - 2x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + x_4 = 0 \end{cases} \quad \text{Set} \begin{cases} x_1 = s \\ x_2 = t \end{cases}$$

$$x = \begin{pmatrix} s \\ t \\ s+2t \\ -2s-2t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

$\Rightarrow$ The vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix}$ form a basis of $U^{\perp}$.

Gram-Schmidt on v_1, v_2:

$$f_1 = \hat{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}.$$

$$w_2 = v_2 - (v_2 \cdot f_1) f_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix} - \frac{1}{6} \cdot 6 \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$f_2 = \hat{w}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, $\underline{f} = (f_1, f_2)$ is an orthonormal basis of $U^{\perp}$.

$$P_{U^\perp}(e_1) = (e_1 \cdot f_1) f_1 + (e_2 \cdot f_2) f_2 = \frac{1}{6}\begin{pmatrix}1\\0\\1\\-2\end{pmatrix} + \frac{-1}{3}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3\\-2\\-1\\-2\end{pmatrix}$$

$$P_{U^\perp}(e_2) = (e_2 \cdot f_1) f_1 + (e_2 \cdot f_2) f_2 = \frac{1}{3}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-2\\2\\2\\0\end{pmatrix}$$

$$P_{U^\perp}(e_3) = \frac{1}{6}\begin{pmatrix}1\\0\\1\\-2\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-1\\2\\3\\-2\end{pmatrix}$$

$$P_{U^\perp}(e_4) = \frac{-2}{6}\begin{pmatrix}1\\0\\1\\-2\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-2\\0\\-2\\4\end{pmatrix}.$$

It follows that $[P_{U^\perp}] = \begin{pmatrix} | & & | \\ P_{U^\perp}(e_1) & \cdots & P_{U^\perp}(e_4) \\ | & & | \end{pmatrix} = \frac{1}{6}\begin{pmatrix}3 & -2 & -1 & -2\\ -2 & 2 & 2 & 0\\ -1 & 2 & 3 & -2\\ -2 & 0 & -2 & 4\end{pmatrix}.$

3.b) By definition, $P_V(x)$ is the unique element in V such that $x - P_V(x) \in V^\perp$.

Now, $x - \underbrace{(x - P_V(x))}_{\in V^\perp} = P_V(x) \in V \implies P_{V^\perp}(x) = x - P_V(x)$.

Hence $P_{V^\perp} = \mathrm{id}_{\mathbb{R}^n} - P_V$, q.e.d.

4.a) Let $R=(r_{ij})_{i,j}$, $S=(s_{ij})_{i,j}$, and $RS=(t_{ij})_{i,j}$.

$R\in T_m$ means that $r_{ij}=0$ for all $i>j$, and $r_{ii}>0$ for all i.

By the definition of matrix multiplication,

$$t_{ij}=\sum_{k=1}^{m} r_{ik}s_{kj} \underset{\uparrow \langle R,S\in T_m\rangle}{=} \sum_{k=i}^{j} r_{ik}s_{kj}$$

Hence: $t_{ij}=\sum_{\emptyset} = 0$ if $i>j$, and
$t_{ii}=r_{ii}s_{ii}>0$ for all i.
$\Big\}\Longrightarrow RS=(t_{ij})_{i,j}\in T_m$.

b) Since the diagonal elements of $R\in T_m$ are non-zero, and R is upper triangular, it follows that $\operatorname{rank}(R)=m$.
$\Longrightarrow$ R is invertible.

Let $R=(r_{ij})_{i,j}\in T_m$ and $R^{-1}=(s_{ij})_{i,j}$.

Suppose that R^{-1} is <u>not</u> upper triangular. Then there exists some $\ell\in\{2,\dots,m\}$ such that the first non-zero element of the ℓ-th row, $s_{\ell k}$, satisfies $\ell>k$. In other words, $s_{\ell k}\neq 0$, and $s_{\ell q}=0$ for all $q<k$.

Let $R^{-1}R=(t_{ij})_{i,j}$. Then
$$t_{\ell k}=\sum_{\mu=0}^{m} s_{\ell\mu}r_{\mu k}=\sum_{\mu=0}^{k-1}\underbrace{s_{\ell\mu}}_{=0}r_{\mu k}+\underbrace{s_{\ell k}r_{kk}}_{\neq 0}+\sum_{\mu=k+1}^{m} s_{\ell\mu}\underbrace{r_{\mu k}}_{=0}\neq 0$$

But $R^{-1}R=I_m$, so $t_{ij}=0$ whenever $i\neq j$.
In particular, $t_{\ell k}=0$, a contradiction. Hence R^{-1} <u>is</u> upper triangular.

As R^{-1} is upper triangular, we have $s_{ij} = 0$ for all $i > j$.

Now, the diagonal elements of $R^{-1}R$ are

$$t_{ii} = \sum_{k=1}^{m} s_{ik} r_{ki} = s_{ii} r_{ii} .$$

On the other hand, $R^{-1}R = I_m$, so $t_{ii} = 1$.

$\Big\}\ s_{ii} r_{ii} = 1$

$s_{ii} = \frac{1}{r_{ii}} > 0$ (as $r_{ii} > 0$)

We have proved that R^{-1} is upper triangular with strictly positive diagonal entries, i.e., $R^{-1} \in T_m$.

c) $I_m = A^T A = (BC)^T BC = C^T \underbrace{B^T B}_{= I_m} C = C^T I_m C = C^T C$.

So $C^T C = I_m$, that is, $C \in O_{m,m}$.

d) If $A = (a_{ij})_{i,j} \in O_{m,m} \cap T_m$ then, in particular, A is orthogonal, so $A^T = A^{-1}$.
By (b), $A^T = A^{-1} \in T_m$.
Hence both A and A^T are upper triangular, which implies that A is a diagonal matrix: $a_{ij} = 0$ whenever $i \neq j$.
Moreover, $A \in T_m \Rightarrow a_{ii} > 0$, $\forall i$.

Now, we have $Ae_i = a_{ii} e_i$, and since $\|Ae_i\| = 1$ and $a_{ii} > 0$ it follows that $a_{ii} = 1$ for all i.
Hence $A = I_m$. This proves that $O_{m,m} \cap T_m \subset \{I_m\}$.
Conversely, it is clear that $I_m \in O_{m,m} \cap T_m$, so

$$O_{m,m} \cap T_m = \{I_m\}$$

e) Assume that $Q_1 R_1 = Q_2 R_2$, where $Q_1, Q_2 \in O_{n,m}$ and $R_1, R_2 \in T_m$.

Then R_1 is invertible and $Q_1 = Q_2 R_2 R_1^{-1}$.

By (c), $R_2 R_1^{-1} \in O_{m,m}$.

By (a) and (b), $R_2 R_1^{-1} \in T_m$

$\overset{(d)}{\Longrightarrow} R_2 R_1^{-1} = I_m$

$\Longrightarrow$ $\underline{\underline{Q_1 = Q \text{ and } R_2 = R_1}}$.