<u>Linear algebra II, spring term 2018</u>

<u>Solutions to Homework 1</u>

1.a) The eq. $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$ has augmented matrix

$$1 \;\; -1 \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 8 \end{pmatrix} \sim 2 \;\; 3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 9 \end{pmatrix} \sim -1 \begin{pmatrix} 1 & 0 & -5 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ u_1, u_2 are linearly independent and $U = \operatorname{span}\{u_1, u_2\}$

$\Rightarrow$ <u>$\underline{u} = (u_1, u_2)$ is a basis of U.</u>

Moreover, $\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$, so $\begin{cases} [u_1]_{\underline{u}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ [u_2]_{\underline{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ [u_3]_{\underline{u}} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \end{cases}$

b) Since $u_i \cdot x = u_i^T x$, it follows that $F(x) = \begin{pmatrix} - u_1^T - \\ - u_2^T - \\ - u_3^T - \end{pmatrix} x$ for all $x \in \mathbb{R}^3$,

that is, $[F] = \begin{pmatrix} - u_1^T - \\ - u_2^T - \\ - u_3^T - \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 8 \end{pmatrix}$.

2.a) Let $u=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$, $v=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\in\mathbb{R}^3$, $\lambda\in\mathbb{R}$.

$$\varphi(u+v)=\varphi\begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix}=(u_1+v_1)+(u_2+v_2)(1-x^2)+(u_3+v_3)x^2$$

$$=u_1+u_2(1-x^2)+u_3x^2+v_1+v_2(1-x^2)+v_3x^2=\varphi(u)+\varphi(v)$$

$$\varphi(\lambda u)=\varphi\begin{pmatrix}\lambda u_1\\\lambda u_2\\\lambda u_3\end{pmatrix}=\lambda u_1+\lambda u_2(1-x^2)+\lambda u_3x^2=\lambda\left(u_1+u_2(1-x^2)+u_3x^2\right)=\lambda\varphi(u)$$

Hence, $\underline{\varphi \text{ is a linear map.}}$

b) $\varphi(u)=u_1+u_2(1-x^2)+u_3x^2=(u_1+u_2)+(-u_2+u_3)x^2$.

Thus, $\varphi(u)=0 \iff \begin{cases}u_1+u_2=0\\-u_2+u_3=0\end{cases} \iff u=t\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ for some $t\in\mathbb{R}$.

$\Rightarrow \ker\varphi=\operatorname{span}\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}$, and the $\underline{\text{single vector } \begin{pmatrix}1\\-1\\-1\end{pmatrix} \text{ forms}}$ $\underline{\text{a basis of } \ker\varphi}$.

$\varphi(u)=(u_1+u_2)+(-u_2+u_3)x^2\in\operatorname{span}\{1,x^2\}\Rightarrow \operatorname{im}\varphi\subset\operatorname{span}\{1,x^2\}$.

$3=\dim\mathbb{R}^3=\dim(\ker\varphi)+\dim(\operatorname{im}\varphi)=1+\dim(\operatorname{im}\varphi)$

$\Rightarrow 2=\dim(\operatorname{im}\varphi)\leq\dim(\operatorname{span}\{1,x^2\})\leq 2 \Rightarrow \dim(\operatorname{im}\varphi)=\dim(\operatorname{span}\{1,x^2\})$

$\Rightarrow \operatorname{im}\varphi=\operatorname{span}\{1,x^2\}$ and $1, x^2$ are linearly independent.

Hence, $\underline{(1,x^2) \text{ is a basis of } \operatorname{im}\varphi}$.

3.a) Let $ad_M : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $ad_M(A) = MA - AM$.

Then $ad_M(A+B) = M(A+B) - (A+B)M = MA + MB - AM - BM = ad_M(A) + ad_M(B)$
and
$$ad_M(\lambda A) = M(\lambda A) - (\lambda A)M = \lambda(MA - AM) = \lambda\, ad_M(A)$$
for all $A, B \in \mathbb{R}^{2\times 2}$, $\lambda \in \mathbb{R}$.

Hence ad_M is a linear map.
Since $C(M) = \ker(ad_M)$, it follows that $\underline{\underline{C(M) \text{ is a subspace of } \mathbb{R}^{2\times 2}}}$

b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $ad_M(A) = MA - AM = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$$= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

so $A \in C(M) = \ker(ad_M) \iff b = c = 0$.

$$\Rightarrow C(M) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, a, d \in \mathbb{R} \right\} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{22}} \right\}$$

$\lambda_1 E_{11} + \lambda_2 E_{22} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = 0 \iff \lambda_1 = \lambda_2 = 0$, so E_{11}, E_{22} are lin. independent.

Hence (E_{11}, E_{22}) is a basis of $C(M)$.

c) Let $F, G \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Then, for all $u, v \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$,

$$(F+G)(u+v) = F(u+v) + G(u+v) = F(u) + F(v) + G(u) + G(v) = (F+G)(u) + (F+G)(v) \text{ , and}$$
$$(F+G)(\lambda u) = F(\lambda u) + G(\lambda u) = \lambda F(u) + \lambda G(u) = \lambda\left(F(u) + G(u)\right) = \lambda (F+G)(u)$$
$$\Longrightarrow F+G \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n).$$

Let $F \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $\lambda \in \mathbb{R}$. Then, for all $u, v \in \mathbb{R}^m$, $\mu \in \mathbb{R}$,

$$(\lambda F)(u+v) = \lambda \cdot F(u+v) = \lambda \cdot (F(u) + F(v)) = \lambda F(u) + \lambda F(v) = (\lambda F)(u) + (\lambda F)(v) \text{ , and}$$
$$(\lambda F)(\mu u) = \lambda \cdot F(\mu u) = \lambda \mu F(u) = \mu \cdot (\lambda F(u)) = \mu (\lambda F)(u)$$
$$\Longrightarrow \lambda F \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

Moreover, the zero function n satisfies

$n(u+v) = 0 = n(u) + n(v)$ and $n(\lambda u) = 0 = \lambda n(u)$ for all $u, v \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$.

Hence, $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \subset \mathcal{T}(\mathbb{R}^m, \mathbb{R}^n)$ is a subspace.

4.a) Let $f(x) = a_1 + a_2 x + a_3 x^2 \in \mathcal{P}_2$.

Then $\left.\begin{aligned} f'(x) = a_2 + 2a_3 x &\Rightarrow f'(0) = a_2 \\ f''(x) = 2a_3 &\Rightarrow f''(0) = 2a_3 \end{aligned}\right\} \Rightarrow d(f) = \begin{pmatrix} a_1 \\ a_2 \\ 2a_3 \end{pmatrix}$

Let $c : \mathbb{R}^3 \to \mathcal{P}_2$ be the function given by $c\begin{pmatrix} r \\ s \\ t \end{pmatrix} = r + sx + \frac{t}{2}x^2$.

Then $dc\begin{pmatrix} r \\ s \\ t \end{pmatrix} = d\left(r + sx + \frac{t}{2}x^2\right) = \begin{pmatrix} r \\ s \\ t \end{pmatrix} \Longrightarrow dc = id_{\mathbb{R}^3}$, and

$cd\left(a_1 + a_2 x + a_3 x^2\right) = c\begin{pmatrix} a_1 \\ a_2 \\ 2a_3 \end{pmatrix} = a_1 + a_2 x + a_3 x^2 \Longrightarrow cd = id_{\mathcal{P}_2}$

Hence d is invertible, and $d^{-1} = c$.

b) $d(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $d(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $d(x^2) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$\Rightarrow [d]_{\underline{x}}^{\underline{e}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.