Linear algebra II, spring term 2018

Solutions to midterm exam 12th June 2018

1.a) A is **not orthogonal**, since $\|Ae_3\| = \left\| \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{pmatrix} \right\| = \frac{4}{\sqrt{6}} \neq 1$.

b) $x \in U^\perp \iff \begin{cases} x \cdot u_1 = 0 \\ x \cdot u_2 = 0 \end{cases} \iff \begin{cases} x \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \\ x \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \end{cases}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

(-1)

Setting $x_3 = t$, the solutions of the system are:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow U^\perp = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$. An orthonormal basis of $U^\perp$ is given by the vector $\widehat{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}} = \underline{\underline{\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}}$

c) As (u_1, u_2) is an orthonormal basis of U,

$$P_U(x) = (x \cdot u_1)u_1 + (x \cdot u_2)u_2 = \frac{1}{3} 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \underline{\underline{\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}.$$

2) First, determine a basis of U:

$$\begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot el. in first and second column.

$\Rightarrow (u_1, u_2)$ is a basis of U.

Apply the Gram-Schmidt algorithm to (u_1, u_2):

$$v_1 = \hat{u}_1 = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}.$$

$$w_2 = u_2 - (u_2 \cdot v_1)v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}(-1+0+0)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$v_2 = \hat{w}_2 = \widehat{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}} = \underline{\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}.$$

$$\underline{(v_1, v_2) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right) \text{ is an orthonormal basis of } U}$$

3) $V(a,b,c)$ is invertible $\iff \det(V(a,b,c)) \neq 0$.

$$\det(V(a,b,c)) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$$

upper triangular

Hence $V(a,b,c)$ is invertible if and only if $(b-a)(c-a)(c-b) \neq 0$

$$\iff \begin{cases} a \neq b \\ b \neq c \\ c \neq a \end{cases}.$$

4a) A linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ is irreducible if and only if it has no 1-dimensional invariant subspaces.

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by angle $\frac{\pi}{2}$, i.e., $[F] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Assume that $U \subset \mathbb{R}^2$ is an F-invariant subspace containing a non-zero vector $u \in U$.

Then $F(u) \neq 0$ and $F(u) \cdot u = 0 \implies u, F(u)$ are linearly indep.
Since U is F-invariant, $F(u) \in U$, and so $U = \mathbb{R}^2$.
Hence the only F-invariant subspaces of $\mathbb{R}^2$ are $\{0\}$ and $\mathbb{R}^2$
$\longrightarrow$ <u>F is irreducible</u>.

Let $G = id_{\mathbb{R}^2}$. Then <u>every</u> 1-dimensional subspace is G-invariant
$\implies$ <u>G is not irreducible.</u>

4b) Let $F: V \to V$ be an irreducible linear map, and assume that $U \subset V$ and $W \subset V$ are F-invariant subspaces such that $V = U + W$, and $U \cap W = \{0\}$.

Since U is F-invariant and F irreducible, either $U = V$ or $U = \{0\}$ holds.
If $U = \{0\}$ then $V = U + W = \{0\} + W$ implies that for every $v \in V$ there exists a $w \in W$ such that $v = 0 + w$, i.e., $v = w$. Hence, $V = W$.

The above proves that either $U = V$ or $W = U$.
$\Longrightarrow$ <u>F is indecomposable.</u>

c) Let $H: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by the matrix

$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. For any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $H(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$.

Hence, if $x_1 \neq 0$, then $x, H(x)$ are linearly independent.
If $x_1 = 0$ then $H(x) = 0$.

$\Longrightarrow \ker H = \operatorname{span}\{e_2\}$.

Note that $\ker H$ is an H-invariant subspace of $\mathbb{R}^2$.
Moreover, if $U \subset \mathbb{R}^2$ is H-invariant and not contained in $\ker H$, then $\exists x \in U : x_1 \neq 0$, and hence $x, H(x)$ are linearly indep.
As $x, H(x) \in U$ this means that $U = \mathbb{R}^2$.

Consequently, the only H-invariant subspaces of $\mathbb{R}^2$ are $\{0\}$, $\ker H$ and $\mathbb{R}^2$.
They are ordered by inclusion: $\{0\} \subset \ker H \subset \mathbb{R}^2$.

Hence, if $U \subset \mathbb{R}^2$, $W \subset \mathbb{R}^2$ are H-invariant and $U + W = \mathbb{R}^2$, then either $U = \mathbb{R}^2$ or $W = \mathbb{R}^2$.

Thus, H is indecomposable.
H is not irreducible, since $\ker H \subset \mathbb{R}^2$ is H-invariant, and $\ker H \neq \{0\}$, $\ker H \neq \mathbb{R}^2$.