Linear algebra II, spring term 2018
Solutions to final exam, 7th August 2018

1.a) The eigenvalues of F are the zeroes of the characteristic polynomial $f_F(\lambda) = \det(F - \lambda \cdot id_{\mathbb{R}^3}) = \begin{vmatrix} 1-\lambda & 1 & -2 \\ 1 & 1-\lambda & 1 \\ -1 & -1 & 2-\lambda \end{vmatrix} =$

$$= \begin{vmatrix} -\lambda & 0 & -\lambda \\ 1 & 1-\lambda & 1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ -1 & -1 & 3-\lambda \end{vmatrix} = (-\lambda)(1-\lambda)(3-\lambda)$$

$\Rightarrow$ The eigenvalues of F are the numbers $\underline{\underline{0, 1, \text{and } 3}}$.

Eigenspaces:

$\underline{\lambda = 0}$: $A - 0I_3 = A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_0(F) = \ker F = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{v_1} \right\}$$

$\underline{\lambda = 1}$: $A - 1 \cdot I_3 = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \mathcal{E}_1(F) = \ker(F - 1 \cdot id) = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}_{v_2} \right\}$$

$\underline{\lambda=3}$: $A-3I_3 = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & 1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & 0 \\ 0 & -3 & 0 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$\sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_3(F) = \ker(F-3\mathrm{id}) = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_3} \right\}$

The vector $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is a basis of $\mathcal{E}_0(F)$,

the vector $v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ is a basis of $\mathcal{E}_1(F)$, and

$\underline{\underline{v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ is a basis of } \mathcal{E}_3(F)}}$.

b) $\underline{\underline{\mathrm{almu}_F(\lambda) = \mathrm{gemu}_F(\lambda) = 1 \text{ for each eigenvalue } \lambda \in \{0,1,3\}}}$.

c) Set $\underline{\underline{S = S^{\underline{e}}_{\underline{v}} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}}}$, where $\underline{v} = (v_1, v_2, v_3)$.

Since v_1, v_2, v_3 are eigenvectors corresponding to different eigenvalues, they are linearly independent, and thus form a basis of $\mathbb{R}^3$. Therefore, the matrix S is invertible.

Setting $\underline{\underline{D = [F]_{\underline{v}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}}}$, we have

$A = [F]_{\underline{e}} = S^{\underline{e}}_{\underline{v}} [F]_{\underline{v}} S^{\underline{v}}_{\underline{e}} = SDS^{-1}$, as required.

d) Since 0 is an eigenvalue of F, $\mathcal{E}_0(F) = \ker F \neq \{0\}$ and thus $\underline{\underline{F \text{ is not invertible}}}$.

2) Define a function $u : \mathbb{R} \to \mathbb{R}^3$ by $u(t) = [x(t)]_{\underline{v}}$ for all $t \in \mathbb{R}$,

where $\underline{v} = (v_1, v_2, v_3)$ is the eigenbasis from problem 1.

Then $u(t) = S^{-1}x(t)$, and $u'(t) = S^{-1}x'(t) = S^{-1}Ax(t) = S^{-1}ASu(t) = Du(t)$.

I.e., $$\begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} 0 \\ u_2(t) \\ 3u_3(t) \end{pmatrix} \Rightarrow \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 e^t \\ c_3 e^{3t} \end{pmatrix}$$

for some $c_1, c_2, c_3 \in \mathbb{R}$.

Moreover, $$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = x(0) = S \cdot u(0) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 + c_3 \\ -c_1 + 2c_2 \\ c_2 - c_3 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{cases} c_1 = -1, \\ c_2 = 0, \\ c_3 = 1, \end{cases} \text{ so } u(t) = \begin{pmatrix} -1 \\ 0 \\ e^{3t} \end{pmatrix}$$

The function $x : \mathbb{R} \to \mathbb{R}^3$ is given by

$$x(t) = Su(t) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ e^{3t} \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1 + e^{3t} \\ 1 \\ -e^{3t} \end{pmatrix}}}, \qquad t \in \mathbb{R}$$

3) Let $T = D^3 - 2D^2 = D^2(D - 2\cdot id)$, so that the equation can be written as $T(f) = 4$.

The characteristic polynomial of T is $p_T(x) = x^2(x-2)$, which means that $(1, t, e^{2t})$ is a basis of $\ker T$.

The general solution to the eq. can now be writen as

$f(t) = c_1 + c_2 t + c_3 e^{2t} + f_p(t)$, where f_p is any one solution to the eq. $T(f) = 4$, and $c_1, c_2, c_3 \in \mathbb{R}$.

Since any linear polynomial $c_1 + c_2 t$ is in $\ker T$, it follows that if the eq. $T(f) = 4$ has a solution that is a quadratic polynomial then it has a solution of the form $f_p(t) = at^2$ for some $a \in \mathbb{R}$.

If $f_p(t) = at^2$ then $f_p''(t) = 2a$, $f_p'''(t) = 0$, $\forall t$.

So $T(f_p) = 4 \iff 0 - 2\cdot 2a = 4 \iff a = -1$.
Hence, $f_p(t) = -t^2$ is a solution to $T(f) = 4$, and the general solution has the form

$$\underline{f(t) = c_1 + c_2 t + c_3 e^{2t} - t^2}.$$

Use the initial value conditions $f(0) = f'(0) = f''(0) = 0$ to determine $c_1, c_2, c_3 \in \mathbb{R}$:

$$f'(t) = c_2 + 2c_3 e^{2t} - 2t, \quad f''(t) = 4c_3 e^{2t} - 2, \text{ so}$$

$$f(0) = c_1 + c_3, \quad f'(0) = c_2 + 2c_3, \quad f''(0) = 4c_3 - 2.$$

$$\begin{cases} f(0) = 0 \\ f'(0) = 0 \\ f''(0) = 0 \end{cases} \iff \begin{cases} c_1 + c_3 = 0 \\ c_2 + 2c_3 = 0 \\ 4c_3 = 2 \end{cases} \cdot \tfrac{1}{4} \iff \begin{cases} c_1 + c_3 = 0 \\ c_2 + 2c_3 = 0 \\ c_3 = 1/2 \end{cases} \ (-1),\ (-2)$$

$$\iff \begin{cases} c_1 = -1/2, \\ c_2 = -1, \\ c_3 = 1/2. \end{cases}$$

Hence, the solution of the differential equation, with the initial value conditions, is

$$\underline{\underline{f(t) = -\frac{1}{2} - t - t^2 + \frac{1}{2}e^{2t}.}}$$

4) A basis of $\mathcal{P}_2$ is $\underline{b} = (f_0, f_1, f_2)$, where $f_0(t) = 1$, $f_1(t) = t$
$f_2(t) = t^2$ for all $t \in \mathbb{R}$.

$$\varphi(f_0)(t) = t f_0'(t) + f_0(t) = f_0(t) \Rightarrow \varphi(f_0) = f_0$$

$$\varphi(f_1)(t) = t f_1'(t) + f_1(t) = 2t = 2 f_1(t) \Rightarrow \varphi(f_1) = 2 f_1$$

$$\varphi(f_2)(t) = t f_2'(t) + f_2(t) = t \cdot 2t + t^2 = 3t^2 = 3 f_2(t) \Rightarrow \varphi(f_2) = 3 f_2 .$$

Hence, $[\varphi]_{\underline{b}} = \begin{pmatrix} | & | & | \\ [\varphi(f_0)]_{\underline{b}} & [\varphi(f_1)]_{\underline{b}} & [\varphi(f_2)]_{\underline{b}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$,

and $\underline{\underline{\det(\varphi) = \det([\varphi]_{\underline{b}}) = 1 \cdot 2 \cdot 3 = 6}}$.

5) If $v \in V$ is an eigenvector of P with eigenvalue $\lambda \in \mathbb{R}$, then

$$\lambda v = Pv = P^2 v = P(\lambda v) = \lambda P v = \lambda^2 v \Rightarrow (\lambda^2 - \lambda) v = 0 \overset{v \neq 0}{\Longrightarrow} \underline{\lambda \in \{0, 1\}}.$$

Hence, P is diagonalisable $\Leftrightarrow \dim(\mathcal{E}_0(P)) + \dim(\mathcal{E}_1(P)) = \dim V$.

For any $w = Pv \in \operatorname{im} P$, we have $Pw = P(Pv) = Pv = w$
$\Rightarrow \operatorname{im} P \subset \mathcal{E}_1(P)$.

Therefore, $\dim(\operatorname{im} P) \leq \dim \mathcal{E}_1(P)$ and, by the Kernel-image theorem:

$$\dim V = \dim(\ker P) + \dim(\operatorname{im} P) \leq \dim(\mathcal{E}_0(P)) + \dim(\mathcal{E}_1(P)) \leq \dim V$$

(note that $\mathcal{E}_0(P) = \ker P$)

$\Rightarrow \dim(\mathcal{E}_0(P)) + \dim(\mathcal{E}_1(P)) = \dim V$, so $\underline{P \text{ is diagonalisable}}$.