NAGOYA UNIVERSITY
Global 30 programme
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**Final exam in Linear Algebra II**
*Date:* 7th August 2018 *10:30 – 12:00*
*Material allowed:* Writing material only.

- This exam consists of five problems, with a total score of 45 points.
- All solutions should include motivations and clear answers to the questions asked.
- According to *Nagoya University student discipline rules*, cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the term.
- *Do not forget to write your name on each piece of paper you hand in!*

**1.** Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $F(x) = Ax$, where $A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$.

a) Determine the eigenvalues of $F$, and bases of their respective eigenspaces.

b) What are the algebraic and geometric multiplicities of the eigenvalues of $F$?

c) Give an invertible matrix $S$ and a diagonal matrix $D$ such that $A = SDS^{-1}$.

d) Is $F$ invertible?

*(12 points)*

**2.** Let $A \in \mathbb{R}^{3\times 3}$ be the same matrix as in the previous problem. Determine a closed formula for the differentiable function $x : \mathbb{R} \to \mathbb{R}^3$ that satisfies

$$\begin{cases} x'(t) = A\,x(t) \qquad \text{for all } t \in \mathbb{R}, \\ x(0) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \end{cases}$$

*(9 points)*

**3.** Solve the differential equation

$$f'''(t) - 2f''(t) = 4$$

with the initial value conditions $f(0) = f'(0) = f''(0) = 0$.

*Hint: The equation, without the initial value conditions, has a solution that is a quadratic polynomial.*

*(12 points)*

**4.** Let $\phi : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map given by $\phi(f)(t) = tf'(t) + f(t)$. Compute the determinant of $\phi$.

*Recall that $\mathcal{P}_2$ denotes the vector space of polynomials of degree at most 2.*

*(7 points)*

**5.** A *projector* on a finite-dimensional vector space $V$ is a linear map $P : V \to V$ satisfying $P^2 = P$. Show that every projector is diagonalisable.
*Hint: What are the possible eigenvalues, and how can their eigenspaces be described?*
*(5 points)*