Linear Algebra 2, spring term 2017
Solutions to Homework 4

1.a) Characteristic polynomial: $p(x) = x^2 - 9 = (x+3)(x-3)$

$$\Longrightarrow \quad f(t) = c_1 e^{-3t} + c_2 e^{3t}, \quad c_1, c_2 \in \mathbb{R}$$

b) Char pol: $p(x) = x^2 - 4x + 4 = (x-2)^2$ Zero: $x = 2$, of multiplicity two.

General solution: $f(t) = c_1 e^{2t} + c_2 t e^{2t}, \quad c_1, c_2 \in \mathbb{R}$.

$$f'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} = (2c_1 + c_2) e^{2t} + 2c_2 t e^{2t}$$

Initial conditions:
$$\left.\begin{array}{l} f(0) = 1 : c_1 = 1 \\ f'(0) = 0 : 2c_1 + c_2 = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} c_1 = 1 \\ c_2 = -2 \end{array}\right.$$

$$\Longrightarrow \quad f(t) = e^{2t} - 2te^{2t}$$

c) Char pol: $p(x) = x^3 - 2x + 1$. By inspection we see that $p(1) = 0$

$$\begin{array}{rl} & x^2 + x - 1 \\ \hline x^3 \quad\;\; - 2x + 1 & \big|\; x - 1 \\ x^3 - x^2 \quad\quad\quad & \\ \hline x^2 - 2x + 1 & \\ x^2 - x \quad\;\; & \\ \hline -x + 1 & \\ -x + 1 & \\ \hline 0 & \end{array}$$

$$p(x) = (x-1)(x^2 + x - 1) = (x-1)\left(\left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - 1\right)$$

$$= (x-1)\left(\left(x + \tfrac{1}{2}\right)^2 - \left(\tfrac{\sqrt{5}}{2}\right)^2\right) = (x-1)\left(x - \left(-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right)\right)\left(x - \left(-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}\right)\right)$$

So $p(x) = 0 \Longleftrightarrow x = 1$ or $x = \underbrace{-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}}_{\lambda_1}$ or $x = \underbrace{-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}}_{\lambda_2}$

General solution:
$$f(t) = c_1 e^{t} + c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t}$$
$$c_1, c_2, c_3 \in \mathbb{R}$$

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2.a) $f'' = 9f$: $T_1 = D^2 - 9\mathbb{1} = (D - 3\mathbb{1})(D + 3\mathbb{1})$

$f'' - 4f' + 4f = 0$: $T_2 = D^2 - 4D + 4\mathbb{1} = (D - 2\mathbb{1})^2$

$f''' - 2f' + f = 0$: $T_3 = D^3 - 2D + \mathbb{1} = (D - \mathbb{1})(D - \lambda_1\mathbb{1})(D - \lambda_2\mathbb{1})$

where $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{5}}{2}$, $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{5}}{2}$

b) $T_1 = D^2 - 9\mathbb{1}$: $\ker(T_1)$ has a basis (e^{-3t}, e^{3t})

$\ker(T_2)$ has a basis (e^{2t}, te^{2t})

$\ker(T_3)$ has a basis $(e^{t}, e^{\lambda_1 t}, e^{\lambda_2 t})$

3a) If $f_p(t) = at^2 + bt + c$ then $f_p'(t) = 2at + b$

$f_p''(t) = 2a$

Hence, f_p is a solution of $(*)$ iff $2a + (2at + b) - 6(at^2 + bt + c) = \frac{1}{3} - 6t^2$

$$-6a\,t^2 + (2a - 6b)t + (2a + b - 6c) = -6t^2 + \frac{1}{3}$$

$$\begin{cases} -6a = -6 \\ 2a - 6b = 0 \\ 2a + b - 6c = \frac{1}{3} \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ -6b = -2 \\ b - 6c = \frac{1}{3} - 2 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = \frac{1}{3} \\ -6c = -2 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = \frac{1}{3} \\ c = \frac{1}{3} \end{cases}$$

$$f_p(t) = t^2 + \frac{1}{3}t + \frac{1}{3}$$

b) The characteristic polynomial of the homogeneous equation

$f''+f'-6f=0$ is $p(x)=x^2+x-6=(x+\frac{1}{2})^2-\frac{1}{4}-6$

$=(x+\frac{1}{2})^2-(\frac{5}{2})^2=(x+3)(x-2)$

$\Rightarrow$ The homogeneous equation has solution $\underline{f_h(t)=c_1e^{-3t}+c_2e^{2t}}$, $c_1,c_2\in\mathbb{R}$

Hence the general solution of $(*)$ is

$\underline{\underline{f(t)=c_1e^{-3t}+c_2e^{2t}+t^2+\frac{1}{3}t+\frac{1}{3}}}$, $c_1,c_2\in\mathbb{R}$

4a) Eigenvalues of A: $f_A(\lambda)=\det(A-\lambda I_2)=\begin{vmatrix}-\lambda & 1\\ 9 & -\lambda\end{vmatrix}=\lambda^2-9$
$=(\lambda+3)(\lambda-3)$

Eigenvalues of A are $\underline{\lambda_1=-3 \text{ and } \lambda_2=3}$

$A-\lambda_1I_2=\begin{pmatrix}3&1\\9&3\end{pmatrix}\sim\begin{pmatrix}3&1\\0&0\end{pmatrix}\Rightarrow \mathcal{E}_{-3}(A)=\ker(A+3I)=\text{span}\left\{\underbrace{\begin{pmatrix}1\\-3\end{pmatrix}}_{v_1}\right\}$

$A-\lambda_2I_2=\begin{pmatrix}-3&1\\9&-3\end{pmatrix}\sim\begin{pmatrix}-3&1\\0&0\end{pmatrix}\Rightarrow \mathcal{E}_{3}(A)=\text{span}\left\{\underbrace{\begin{pmatrix}1\\3\end{pmatrix}}_{v_2}\right\}$

Set $\underline{v}=(v_1,v_2)$: eigenbasis of $\mathbb{R}^2$.

Now $A=SDS^{-1}$, where $D=\begin{pmatrix}-3&0\\0&3\end{pmatrix}$ and $S=S_{\underline{v}}^{\underline{e}}=\begin{pmatrix}1&1\\-3&3\end{pmatrix}$

Set $u(t)=S^{-1}x(t)$. Then $u'(t)=S^{-1}x'(t)=S^{-1}Ax(t)=DS^{-1}x(t)=Du(t)$.

$\Rightarrow u'(t)=\begin{pmatrix}u_1'(t)\\u_2'(t)\end{pmatrix}=\begin{pmatrix}-3&0\\0&3\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}\Rightarrow u(t)=\begin{pmatrix}c_1e^{-3t}\\c_2e^{3t}\end{pmatrix}$, $c_1,c_2\in\mathbb{R}$

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$$x(t) = Su(t) = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{3t} \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 e^{3t} \\ -3c_1 e^{-3t} + 3c_2 e^{3t} \end{pmatrix}; \quad c_1, c_2 \in \mathbb{R}$$

b) $B = A = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}$

c) f satisfies the equation $f'' = 9f$ if and only if

$x(t) = \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}$ satisfies $x'(t) = Ax(t)$

$$\Longleftrightarrow x(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 e^{3t} \\ -3c_1 e^{-3t} + 3c_2 e^{3t} \end{pmatrix} \Longleftrightarrow f(t) = c_1 e^{-3t} + c_2 e^{3t} \quad (c_1, c_2 \in \mathbb{R})$$

(as in problem 1a)

5a) $D(f_c)(t) = f_c'(t) = ae^{at}\cos(bt) - be^{at}\sin(bt) = af_c(t) - bf_s(t)$

$(D - a\mathbb{1})(f_c)(t) = f_c'(t) - af_c(t) = -b f_s(t)$

$D(f_s)(t) = a e^{at}\sin(bt) + be^{at}\cos(bt) = af_s(t) + bf_c(t)$

$(D - a\mathbb{1})(f_s) = bf_c$

Hence: $(D - a\mathbb{1})^2(f_c) = (D - a\mathbb{1})(-bf_s) = -b(D - a\mathbb{1})(f_s) = -b^2 f_c$

$$\Rightarrow \underline{\underline{T(f_c)}} = (D - a\mathbb{1})^2(f_c) + b^2 f_c = -b^2 f_c + b^2 f_c \underline{\underline{= 0}}$$

and

$(D - a\mathbb{1})^2(f_s) = (D - a\mathbb{1})(bf_c) = b(D - a\mathbb{1})(f_c) = -b^2 f_s$

$$\Rightarrow \underline{\underline{T(f_s)}} = -b^2 f_s + b^2 f_s \underline{\underline{= 0}}$$

b) Assume that $\lambda f_c + \mu f_s = 0$, i.e., $\lambda f_c(t) + \mu f_s(t) = 0$ for all $t \in \mathbb{R}$

$\Rightarrow 0 = \lambda f_c(0) + \mu f_s(0) = \lambda \cdot 1 + \mu \cdot 0 = \lambda$ and

$0 = \lambda f_c\left(\frac{\pi}{2b}\right) + \mu f_s\left(\frac{\pi}{2b}\right) = \lambda e^{\frac{a\pi}{2b}}\cos\left(\frac{\pi}{2}\right) + \mu e^{\frac{a\pi}{2b}}\sin\left(\frac{\pi}{2}\right) = \mu e^{\frac{a\pi}{2b}}$

$\Rightarrow \mu = 0$

Hence, $\lambda = \mu = 0$, implying that f_c, f_s are linearly independent.

c) As T is a differential operator of order 2, we know that $\dim(\ker T)=2$

Moreover, $f_c, f_s \in \ker T$, and are linearly independant.

$\Longrightarrow$ (f_c, f_s) is a basis of $\ker T$

6a) Clearly, if $v \in \ker(FG)$ then $F(\tilde{G}(v)) = F(G(v)) = 0$, so $\tilde{G}(v) \in \ker F$.
This proves that the map $\tilde{G}: \ker FG \to \ker F$ is well defined.

Let $w \in \ker F$. Since $\operatorname{im} G = V \supset \ker F \ni w$, we have $w \in \operatorname{im} G$,
and therefore $w = G(v)$ for some $v \in V$.
Moreover, $FG(v) = F(w) = 0 \Rightarrow v \in \ker(FG)$
Hence, $\tilde{G}(v) = w$ with $v \in \ker(FG)$, that is, $w \in \operatorname{im} \tilde{G}$.
$\Longrightarrow \operatorname{im} \tilde{G} = \ker F$.

b) To show this, we need that if $v \in \ker G$ then $v \in \ker(FG)$.
But if $v \in \ker G$ then $FG(v) = F(G(v)) = F(0) = 0 \Rightarrow v \in \ker(FG)$.
Clearly $\tilde{G}(v) = G(v) = 0 \Rightarrow v \in \ker \tilde{G}$.
This shows that $\ker G \subset \ker \tilde{G}$.
The inclusion $\ker \tilde{G} \subset \ker G$ is immediate: $v \in \ker \tilde{G} \Rightarrow G(v) = \tilde{G}(v) = 0$
$\Rightarrow v \in \ker G$.

Hence, $\ker \tilde{G} = \ker G$.

c) By the kernel-image theorem applied to the map $\tilde{G}: \ker(FG) \to \ker(F)$,

$$\dim(\ker(FG)) = \dim(\ker \tilde{G}) + \dim(\operatorname{im} \tilde{G}) \overset{a,b}{=} \dim(\ker G) + \dim(\ker F)$$

d) • Let $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Setting $f(t) = e^{at}\int e^{-at} g(t)\,dt$,

we get $T(f) = (D - a\mathbb{1})(f) = f' - af = g$ by (iii), that is, $g \in \operatorname{im} T$.

Hence $\underline{\operatorname{im} T = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})}$.

• By (iii), the equation $T(f) = 0$ has solution $f(t) = e^{at}\int 0\,dt = ce^{at}$, $c \in \mathbb{R}$.

This means that $\ker T = \operatorname{span}\{e^{at}\}$, in particular, $\underline{\dim(\ker T) = 1}$.

e) For $n=1$, $T = D - a_1\mathbb{1} \Rightarrow \dim(\ker T) = 1$ by (d).

For $n=2$, $T = \underbrace{(D - a_2\mathbb{1})}_{T_2}\underbrace{(D - a_1\mathbb{1})}_{T_1}$. By (d), $\dim(\ker T_1) = \dim(\ker T_2) = 1$ and $\operatorname{im} T_1 = \operatorname{im} T_2 = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$.

Hence, by (c), $\dim(\ker T) = \dim(\ker(T_1 T_2)) = \dim(\ker T_1) + \dim(\ker T_2) = 2$.
Moreover, since $\operatorname{im} T_1 = \operatorname{im} T_2 = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, it follows that $\operatorname{im}(T) = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$

Continuing this process, we get that for $T = (D - a_n\mathbb{1})\cdots(D - a_1\mathbb{1})$,

$$\dim(\ker T) = \overbrace{\dim(\ker(D - a_n\mathbb{1}))}^{1} + \overbrace{\dim\big(\ker((D - a_{n-1}\mathbb{1})\cdots(D - a_1\mathbb{1}))\big)}^{n-1} = 1 + (n-1) = \underline{\underline{n}}$$

Formally, we argue by induction: For $i = 1,\dots,n$, set $T_i = (D - a_i\mathbb{1})\cdots(D - a_1\mathbb{1})$.

Base case:

$\dim(\ker T_1) = 1$ and $\operatorname{im} T_1 = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ as above.

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Induction hypothesis: Assume that T_{i-1} satisfies $\begin{cases} \dim(\ker T_{i-1}) = 1 \\ \operatorname{im} T_{i-1} = \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \end{cases}$

$(2 \leq i \leq n)$

Induction step:

We have $T_i = (D - a_i \mathbb{1}) T_{i-1}$. By (c),

$$\dim(\ker T_i) = \dim(\ker(D - a_i \mathbb{1})) + \dim T_{i-1} = 1 + (i-1) \quad \left[\text{by the induction hypothesis}\right]$$
$$= i$$

Moreover, $\operatorname{im}(D - a_i \mathbb{1}) = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, and,
by IH, $\operatorname{im} T_{i-1} = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$.

$$\Longrightarrow \operatorname{im}(T) = \operatorname{im}((D - a_i \mathbb{1}) T_{i-1}) = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$$

By induction, it follows that $\dim(\ker T) = n$
(and $\operatorname{im} T = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$)