Linear Algebra II, spring term 2017

Homework 3: Eigenvalues and eigenvectors

Solutions are to be handed in by 14:45 on Thursday the 29th June at latest. Late submissions will, in principle, not be marked.

1. Determine all eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.
Is A diagonalisable? If it is, determine a diagonal matrix B and a matrix S such that $B = S^{-1}AS$.

2. Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be the linear map given by the formula $T(f)(t) = tf'(t)$. Determine all eigenvalues and eigenvectors of T. Is T diagonalisable? If it is, give a basis $\underline{p}$ of $\mathcal{P}_n$ such that the matrix $[T]_{\underline{p}}$ is diagonal.
Recall that $\mathcal{P}_n$ denotes the vector space of all polynomials of degree at most n.

3. Let $A, B, S \in \mathbb{R}^{n\times n}$, with S invertible, and $B = SAS^{-1}$. Given an eigenvalue λ of A,

a) show that $\mathcal{E}_\lambda(B) = \{Sx \in \mathbb{R}^n \mid x \in \mathcal{E}_\lambda(A)\}$;

b) conclude that $\operatorname{gemu}_B(\lambda) = \operatorname{gemu}_A(\lambda)$.

Recall that $\operatorname{gemu}_A(\lambda)$ *denotes the geometric multiplicity of λ in A; that is,* $\operatorname{gemu}_A(\lambda) = \dim \mathcal{E}_\lambda(A) = \dim \ker(A - \lambda \mathrm{I}_n)$.

4. The infinitely (well, almost) large Swedish taiga is inhabitated by many different animals, including wolves and mooses. Wolves eat mooses and mooses eat pine sprouts, the latter of which exist in almost unlimited quantities. The following is a (very simplified) mathematical model for the population sizes of wolf and moose in relation to each other:

Denote by $w(t)$ the size of the wolf population in year t, and by $m(t)$ the size of the moose population in year t. Then the population sizes in the next year are determined by the following formulae:

$$m(t+1) = \frac{4}{3}m(t) - \frac{1}{2}w(t),$$

$$w(t+1) = \frac{1}{6}m(t) + \frac{2}{3}w(t).$$

Denote by $x(t)$ the state vector of the system at time t, i.e., $x(t) = \begin{pmatrix} m(t) \\ w(t) \end{pmatrix}$.

a) Find a 2×2-matrix A such that $x(t+1) = Ax(t)$.

b) Determine the eigenvalues and eigenvectors of the matrix A.

c) Diagonalise A, that is, find matrices B and S such that B is diagonal, S is invertible and $A = SBS^{-1}$.

d) Assume that at a given point in time, there are twice as many wolves as mooses. How will each of the two populations develop in the long term? (Increase/decrease/reach a stable state?)

e) How will the two populations develop if there are twice as many mooses as wolves?