Linear algebra 2, spring term 2017
Solutions to homework assignment 3

1) $f_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ 1 & 3-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$= (2-\lambda)\left((3-\lambda)^2 - 1\right) = (2-\lambda)(2-\lambda)(4-\lambda) = (2-\lambda)^2(4-\lambda)$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$.

Eigenspaces:

$\lambda_1 = 2$: $A - \lambda_1 I_3 = \begin{array}{c} \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}\begin{array}{c} (-1) \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \mathcal{E}_2(A) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$

$\lambda_2 = 4$: $A - \lambda_2 I_3 = \begin{array}{c} (-1) \\ \\ (-\frac{1}{2}) \end{array}\begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \mathcal{E}_4(A) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$

As $\dim \mathcal{E}_2(A) + \dim \mathcal{E}_4(A) = 2 < 3$, the matrix A is not diagonalisable.

2) Let $f(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathcal{P}_n$. Then

$$f'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots + n a_n t^{n-1},$$ so

$$T(f)(t) = t f'(t) = a_1 t + 2a_2 t^2 + \cdots + n a_n t^n.$$

It follows that if $f_i(t) = t^i$ then $T(f_i)(t) = i t^i$, i.e.,

$T(f_i) = i f_i$. Hence f_i is an eigenvector of T with eigenvalue i, for each $i = 0, 1, \ldots, n$.

Since $\dim \mathcal{P}_n = n+1$, there can be at most $n+1$ different eigenvalues of T, so the numbers $0, 1, 2, \ldots, n$ are all the eigenvalues of T.

The sum of the geometric multiplicities of all eigenvalues is always $\leq$ the dimension of the space, with "=" if and only if the map is diagonalisable. It follows that $\mathrm{gemu}_T(i) = \dim \mathcal{E}_i(T) = 1$ for all $i = 0, \ldots, n$, and hence that

$\mathcal{E}_i(T) = \mathrm{span}\{f_i\}$ for all $i = 0, \ldots, n$.

Moreover, $\underline{f} = (f_0, f_1, \ldots, f_n)$ is a basis of $\mathcal{P}_n$ for which

$$[T]_{\underline{f}} = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 2 & \\ 0 & & & \ddots \; n \end{pmatrix}$$

3a) $v \in \mathcal{E}_\lambda(B) \iff Bv = \lambda v \iff SAS^{-1}v = \lambda v \iff AS^{-1}v = \lambda S^{-1}v$

$\iff S^{-1}v \in \mathcal{E}_\lambda(A) \iff v = Sx$ for some $x \in \mathcal{E}_\lambda(A)$

b) Let $(v_1, \dots, v_\ell)$ be a basis of $\mathcal{E}_\lambda(A)$.
Then $(Sv_1, \dots, Sv_\ell)$ is a basis of $\mathcal{E}_\lambda(B)$, since S is invertible.
Hence, $\dim \mathcal{E}_\lambda(B) = \ell = \dim \mathcal{E}_\lambda(A)$, that is, $\mathrm{gemu}_A(\lambda) = \mathrm{gemu}_B(\lambda)$

4a) $A = \begin{pmatrix} 4/3 & -1/2 \\ 1/6 & 2/3 \end{pmatrix}$ satisfies $x(t+1) = Ax(t)$

b) $f_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 4/3 - \lambda & -1/2 \\ 1/6 & 2/3 - \lambda \end{vmatrix} = \left(\frac{1}{6}\right)^2 \cdot \begin{vmatrix} 8 - 6\lambda & -3 \\ 1 & 4 - 6\lambda \end{vmatrix}$

Set $\mu = 6\lambda$

$= \frac{1}{6^2}\left((8-6\lambda)(4-6\lambda) + 3\right) = \frac{1}{6^2}\left((8-\mu)(4-\mu)+3\right) = \frac{1}{6^2}\left(\mu^2 - 12\mu + 35\right)$

$= \frac{1}{6^2}\left((\mu-6)^2 - 36 + 35\right) = \frac{1}{6^2}\left((\mu-6)^2 - 1\right) = \frac{1}{6^2}(\mu-5)(\mu-7) = \frac{1}{6^2}(6\lambda-5)(6\lambda-7)$

Eigenvalues of A are $\lambda_1 = \frac{5}{6}$ and $\lambda_2 = \frac{7}{6}$

Eigenvectors:

$\lambda_1 = \frac{5}{6}$: $A - \lambda_1 I_2 = \begin{pmatrix} 3/6 & -3/6 \\ 1/6 & -1/6 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_{5/6}(A) = \mathrm{span}\left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$

$\lambda_2 = \frac{7}{6}$: $A - \lambda_2 I_2 = \begin{pmatrix} 1/6 & -3/6 \\ 1/6 & -3/6 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_{7/6}(A) = \mathrm{span}\left\{ \underset{v_2}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} \right\}$

c) Setting $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, we get an eigenbasis $\underline{v} = (v_1, v_2)$ of A.

$\underline{S = S_{\underline{v}}^{\underline{e}} = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}}$, $\underline{B = [A]_{\underline{v}} = \begin{pmatrix} 5/6 & 0 \\ 0 & 7/6 \end{pmatrix}}$ gives

$\underline{A = S_{\underline{v}}^{\underline{e}} [A]_{\underline{v}} S_{\underline{e}}^{\underline{v}} = SBS^{-1}}$ as required.

d) In general, $x(t) = A^t x(0)$ for $t \geq 0$, and $A^t = (SBS^{-1})^t = SB^tS^{-1}$

$$B^t = \begin{pmatrix} 5/6 & 0 \\ 0 & 7/6 \end{pmatrix}^t = \begin{pmatrix} (5/6)^t & 0 \\ 0 & (7/6)^t \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix}$$

If $x(0) = \begin{pmatrix} m(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} q \\ 2q \end{pmatrix}$ for some $q > 0$, then

$$x(t) = SB^tS^{-1} \begin{pmatrix} q \\ 2q \end{pmatrix} = \frac{q}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (5/6)^t & 0 \\ 0 & (7/6)^t \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{q}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (5/6)^t & 0 \\ 0 & (7/6)^t \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{q}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \cdot (5/6)^t \\ -(7/6)^t \end{pmatrix} = \frac{q}{2} \begin{pmatrix} 5 \cdot (5/6)^t - 3 \cdot (7/6)^t \\ 5 \cdot (5/6)^t - (7/6)^t \end{pmatrix}$$

As $\frac{5}{6} < 1$, $\left(\frac{5}{6}\right)^t$ decreases as t increases,
while $\left(\frac{7}{6}\right)^t$ increases with t.

Hence, both components of $x(t)$ will decrease as t increases.

(Actually, $m(t)$ will become negative for $t \geq 2$, at which point the model obviously is no longer accurate.)

e) If $x(0) = \begin{pmatrix} m(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} 2q \\ q \end{pmatrix}$ for some $q > 0$, then

$$x(t) = SB^tS^{-1} \cdot x(0) = \frac{q}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (5/6)^t & 0 \\ 0 & (7/6)^t \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{q}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (5/6)^t & 0 \\ 0 & (7/6)^t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{q}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (5/6)^t \\ (7/6)^t \end{pmatrix} = \frac{q}{2} \begin{pmatrix} (5/6)^t + 3 \cdot (7/6)^t \\ (5/6)^t + (7/6)^t \end{pmatrix}$$

Clearly, both $m(t)$ and $w(t)$ will be ultimately increasing, for large enough t.