Linear Algebra 2, spring term 2017
Solutions to Homework 2

1a) Clearly $u_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$ form a basis of U, whilst $u_3 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ forms a basis of $U^\perp$. By Gram-Schmidt, or simple inspection, one finds that $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix}$ form an orthonormal basis of U.

Hence $\underline{v} = (v_1, v_2, v_3)$, where $v_3 = \hat{u}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$ is an orthonormal basis of $\mathbb{R}^3$ s.th. $v_1, v_2 \in U$.

b) For example, $A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$. This matrix is orthogonal since its columns form an orthonormal basis of $\mathbb{R}^3$, and $Ae_1 = v_1$, $Ae_2 = v_2 \in U$.

2) The least-square solutions are given by the solutions of the equation $A^TAx = A^Tb$.

$$A^TA = \begin{pmatrix}1 & 0 & 1\\1 & 2 & 2\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\0 & 2 & 0\\1 & 2 & 1\end{pmatrix} = \begin{pmatrix}2 & 3 & 2\\3 & 9 & 3\\2 & 3 & 2\end{pmatrix}. \quad A^Tb = \begin{pmatrix}1 & 0 & 1\\1 & 2 & 2\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\5\\2\end{pmatrix}$$

The eq. $A^TAx = A^Tb$ has augm. matrix

$$\left(\begin{array}{ccc|c}2 & 3 & 2 & 2\\3 & 9 & 3 & 5\\2 & 3 & 2 & 2\end{array}\right) \begin{array}{c}-1\\ \cdot\frac{1}{3}\\ \end{array} \sim \left(\begin{array}{ccc|c}2 & 3 & 2 & 2\\1 & 3 & 1 & 5/3\\0 & 0 & 0 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}0 & -3 & 0 & -4/3\\1 & 3 & 1 & 5/3\\0 & 0 & 0 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}0 & -3 & 0 & -4/3\\1 & 0 & 1 & 1/3\\0 & 0 & 0 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}0 & 1 & 0 & 4/9\\1 & 0 & 1 & 1/3\\0 & 0 & 0 & 0\end{array}\right)$$

$$\begin{cases} x_2 = -\frac{4}{9} \\ x_1 + x_3 = 1/3 \end{cases} \quad \text{Set } x_3 = t. \qquad x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1/3 - t\\4/9\\t\end{pmatrix} = \begin{pmatrix}1/3\\4/9\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad t \in \mathbb{R}$$

are the least-square solutions to the eq. $Ax = b$

3) $$\det A = \left(\frac{1}{2}\right)^3 \begin{vmatrix} 2 & -2 & 0 \\ 1 & 2 & 3 \\ -3 & 0 & 3 \end{vmatrix} = \frac{1}{8}\cdot 2\cdot 3 \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{vmatrix} =$$

$$= \frac{3}{4}\left(1\cdot 2\cdot 1 + (-1)\cdot 3\cdot(-1) + 0\cdot 1\cdot 0 - (-1)\cdot 2\cdot 0 - 0\cdot 3\cdot 1 - 1\cdot 1\cdot(-1)\right)$$

$$= \frac{3}{4}\left(2+3-(-1)\right) = \frac{3}{4}\cdot 6 = \underline{\underline{\frac{9}{2}}}$$

$$\det B = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 5 \\ 0 & 1 & 2 & -6 \\ 2 & 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 6 \\ 0 & 1 & 2 & -6 \\ 0 & -1 & -2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 6 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1\cdot 1\cdot(-2)\cdot 1 = \underline{\underline{2}}$$

4a) The matrix $A_t = A - tI_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} - \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} 1-t & 1 & 1 \\ 0 & 2-t & 0 \\ 1 & 2 & 1-t \end{pmatrix}$

is invertible if and only if its determinant is non-zero.

$$\det A_t = \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 2-t & 0 \\ 1 & 2 & 1-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & 1 \\ 1 & 1-t \end{vmatrix}$$

$$= (2-t)\left((1-t)^2 - 1^2\right) = (2-t)(1-t-1)(1-t+1) = (2-t)(-t)(2-t) = (2-t)^2(-t)$$

Hence, $\det A_t \neq 0 \iff t \neq 2$ and $t \neq 0$.

A_t is invertible for all $t \neq 0, 2$.

b) $Ax = \lambda x \iff Ax - \lambda x = 0 \iff (\underbrace{A - \lambda I_3}_{= A_\lambda})x = 0 \iff x \in \ker(A_\lambda)$

The matrix A_t is invertible for all $t \neq 0, 2$, so in these cases $\ker A_t = \{0\}$.

If $\lambda = 0$: $Ax = 0x \iff x \in \ker A$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \ker A = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

S_0: $Ax = 0x \iff x = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}$

If $\lambda = 2$: $Ax = 2x \iff x \in \ker A_2$

$$A_2 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \ker A_2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Hence $Ax = 2x \iff x \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

c) $x \in \mathcal{E}_\lambda(B) \iff Bx = \lambda x \iff Bx - \lambda x = 0 \iff (B - \lambda I_n)x = 0$

$\iff x \in \ker(B - \lambda I_n)$.

Hence $\mathcal{E}_\lambda(B) = \ker(B - \lambda I_n) \subset \mathbb{R}^n$ — a subspace