Linear Algebra II, spring term 2017
Solutions to Homework 1

1a) Use the Gram-Schmidt algorithm.

$v_1 = \hat{u}_1 = \frac{1}{\|u_1\|} u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\|u_1\| = \sqrt{2}$

$w_2 = u_2^\perp = u_2 - (u_2 \cdot v_1) v_1$ $\qquad u_2 \cdot v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$

$w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$. $\|w_2\| = \frac{1}{2}\sqrt{6} = \sqrt{\frac{3}{2}}$

$v_2 = \hat{w}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

$w_3 = u_3^\perp = u_3 - (u_3 \cdot v_1) v_1 - (u_3 \cdot v_2) v_2$

$u_3 \cdot v_1 = 0$, $\quad u_3 \cdot v_2 = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

$w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$

$\|w_3\| = \frac{1}{3}\sqrt{12} = 2/\sqrt{3}$

$v_3 = \hat{w}_3 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$

$\underline{v} = (v_1, v_2, v_3)$, where $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, $v_3 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$ is an orthonormal basis of U.

1b) The QR factorisation of $A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$ is

$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} \|u_1\| & u_2 \cdot v_1 & u_3 \cdot v_1 \\ 0 & \|w_2\| & u_3 \cdot v_2 \\ 0 & 0 & \|w_3\| \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{3/2} & \sqrt{2/3} \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}}}$$

2a) $P_U(x) = (x \cdot v_1)v_1 + (x \cdot v_2)v_2 + (x \cdot v_3)v_3$, since v_1, v_2, v_3 is an orthonormal basis of U. Hence:

$$P_U(e_1) = \frac{1}{\sqrt{2}}v_1 + \left(-\frac{1}{\sqrt{6}}\right)v_2 + \frac{1}{\sqrt{12}}v_3 = \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{1}{6}\begin{pmatrix}-1\\1\\2\\0\end{pmatrix} + \frac{1}{12}\begin{pmatrix}1\\-1\\1\\3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3\\1\\-1\\1\end{pmatrix}$$

$$P_U(e_2) = \frac{1}{\sqrt{2}}v_1 + \frac{1}{\sqrt{6}}v_2 + \left(-\frac{1}{\sqrt{12}}\right)v_3 = \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-1\\1\\2\\0\end{pmatrix} - \frac{1}{12}\begin{pmatrix}1\\-1\\1\\3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1\\3\\1\\-1\end{pmatrix},$$

$$P_U(e_3) = 0v_1 + \frac{2}{\sqrt{6}}v_2 + \frac{1}{\sqrt{12}}v_3 = \frac{1}{3}\begin{pmatrix}-1\\1\\2\\0\end{pmatrix} + \frac{1}{12}\begin{pmatrix}1\\-1\\1\\3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-1\\1\\3\\1\end{pmatrix},$$

$$P_U(e_4) = 0v_1 + 0v_2 + \frac{3}{\sqrt{12}}v_3 = \frac{1}{4}\begin{pmatrix}1\\-1\\1\\3\end{pmatrix}.$$

Consequently, $$[P_U] = \begin{pmatrix} | & | & | & | \\ P_U(e_1) & P_U(e_2) & P_U(e_3) & P_U(e_4) \\ | & | & | & | \end{pmatrix} = \underline{\underline{\frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{pmatrix}}}$$

2b) Every vector $x \in \mathbb{R}^4$ can be written as $x = u + v$ for some $u \in U$ and $v \in U^\perp$. It follows that $u = P_U(x)$ and $v = P_{U^\perp}(x)$, so $x = P_U(x) + P_{U^\perp}(x)$ and hence $P_{U^\perp}(x) = x - P_U(x) = (id_{\mathbb{R}^4} - P_U)(x)$

Therefore, $[P_{U^\perp}] = [id_{\mathbb{R}^4} - P_U] = [id_{\mathbb{R}^4}] - [P_U] = I_4 - [P_U]$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \underline{\underline{\frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}}}$$

3) Map φ is linear:

Let $f, g \in \mathcal{P}_2$. $\varphi(f+g) = \begin{pmatrix} (f+g)(-1) \\ (f+g)(0) \\ (f+g)(1) \end{pmatrix} = \begin{pmatrix} f(-1) + g(-1) \\ f(0) + g(0) \\ f(1) + g(1) \end{pmatrix} = \begin{pmatrix} f(-1) \\ f(0) \\ f(1) \end{pmatrix} + \begin{pmatrix} g(-1) \\ g(0) \\ g(1) \end{pmatrix} = \varphi(f) + \varphi(g)$

Let $f \in \mathcal{P}_2$, $\lambda \in \mathbb{R}$.

$\varphi(\lambda f) = \begin{pmatrix} (\lambda f)(-1) \\ (\lambda f)(0) \\ (\lambda f)(1) \end{pmatrix} = \begin{pmatrix} \lambda \cdot f(-1) \\ \lambda \cdot f(0) \\ \lambda \cdot f(1) \end{pmatrix} = \lambda \begin{pmatrix} f(-1) \\ f(0) \\ f(1) \end{pmatrix} = \lambda \varphi(f)$. <u>Hence φ is linear.</u>

φ invertible: Compute $\ker\varphi$: Assume that $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$ and that $f \in \ker\varphi$. This means that $\begin{pmatrix} f(-1) \\ f(0) \\ f(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, i.e., $\begin{cases} a + b\cdot(-1) + c\cdot(-1)^2 = 0 \\ a + b\cdot 0 + c \cdot 0^2 = 0 \\ a + b \cdot 1 + c \cdot 1^2 = 0 \end{cases}$

$\iff \begin{cases} a - b + c = 0 \\ a = 0 \\ a + b + c = 0 \end{cases} \iff \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} \iff f = 0$. Hence $\ker\varphi = \{0\}$.

Sine $\dim \mathbb{R}^3 = \dim \mathcal{P}_2 = 3$, it follows that <u>$\varphi$ is invertible.</u>

4) $x \in U^{\perp} \iff x \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0$ and $x \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$

$$\iff \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 - x_3 + x_4 = 0 \end{cases} \iff \begin{cases} x_1 + x_2 = 0 \\ -2x_2 - x_3 + x_4 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 + x_2 = 0 \\ x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \end{cases} \iff \begin{cases} x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_4 = 0 \\ x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \end{cases}$$

Set $x_3 = s$, $x_4 = t$.
$$\begin{cases} x_1 = \frac{1}{2}s - \frac{1}{2}t \\ x_2 = -\frac{1}{2}s + \frac{1}{2}t \\ x_3 = s \\ x_4 = t \end{cases} \quad x = \frac{1}{2}s \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2}t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

Hence, $U^{\perp} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}}_{u_2} \right\}$

The vectors u_1, u_2 are linearly independent, but not orthogonal to each other. To produce an orthonormal basis of $U^{\perp}$, we use Gram-Schmidt:

$$v_1 = \hat{u}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$w_2 = u_2 - (u_2 \cdot v_1) v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \left(\frac{-2}{\sqrt{6}} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 2 \\ 6 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

$$v_2 = \hat{w}_2 = \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

The vectors $v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ form an orthonormal basis of $U^{\perp}$.