Linear Algebra II, spring term 2017

Solutions to mid-term examination the 30th May 2017

1) A matrix is invertible if and only if its determinant is non-zero.

$\det A = a^2+1 \neq 0$ for all $a \in \mathbb{R} \Rightarrow$ A is invertible for all $a \in \mathbb{R}$.

$$\det B = \begin{vmatrix} 1 & b & 2 & 1 \\ b & 1 & -3 & 1 \\ 0 & 0 & 1 & b \\ 0 & 0 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & b & 2 & 1 \\ 0 & 1-b^2 & -3-2b & 1-b \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 7-b \end{vmatrix} = 1\cdot(1-b^2)\cdot 1\cdot(7-b) = (1+b)(1-b)(7-b)$$

$\det B = 0 \Leftrightarrow b = \pm 1$ or $b = 7$.

$\Rightarrow$ B is invertible for all $b \in \mathbb{R}$ except $b = \pm 1$ and $b = 7$

2) A matrix is orthogonal if and only if its columns form an orthonormal basis.

$\|Ae_1\| = \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 1$. Set $v = Ae_2$. Then A is orthogonal iff $\|v\| = 1$ and $v \cdot Ae_1 = 0$.

$v \cdot Ae_1 = 0 \Leftrightarrow v \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \Leftrightarrow v_1 = -v_2$ $\quad \left(v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$

$\Rightarrow v = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for some $t \in \mathbb{R}$.

Now $\|v\| = |t|\sqrt{2}$, so $\|v\| = 1 \Leftrightarrow |t| = \frac{1}{\sqrt{2}} \Leftrightarrow t = \pm \frac{1}{\sqrt{2}}$.

Setting $s = \sqrt{2}\,t$, we get

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & s \\ 1 & -s \end{pmatrix}, \text{ where } s \in \{-1, 1\}$$

3) Perform Gram–Schmidt on the vectors $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$ spanning $\mathcal{U}$:

Step 1: $v_1 = \hat{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Step 2: $w_2 = u_2 - P_{v_1}(u_2) = u_2 - (u_2 \cdot v_1)\, v_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

$v_2 = \hat{w}_2 = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

Step 3: $w_3 = u_3 - P_{\mathrm{span}\{v_1, v_2\}}(u_3) = u_3 - (u_3 \cdot v_1) v_1 - (u_3 \cdot v_2) v_2 =$

$u_3 \cdot v_1 = \frac{1}{\sqrt{2}} \cdot 4 = 2\sqrt{2}$. $\quad u_3 \cdot v_2 = \frac{1}{\sqrt{11}} \cdot (-11) = -\sqrt{11}$

$= \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} - 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left(-\sqrt{11}\right) \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = 0$

Hence $u_3 - P_{\mathrm{span}\{v_1, v_2\}}(u_3) = 0$, that is, $u_3 \in \mathrm{span}\{v_1, v_2\}$,

$\Longrightarrow \mathcal{U} = \mathrm{span}\{v_1, v_2\}$, and $\underline{v} = (v_1, v_2)$ is an $\overbrace{\ }^{\text{orthonormal}}$ basis of $\mathcal{U}$.

Basis of $\mathcal{U}^\perp$: $x \in \mathcal{U}^\perp \Longleftrightarrow x \cdot v_1 = x \cdot v_2 = 0 \Longleftrightarrow x \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = 0$

Augm. matr.: $\begin{array}{c} (-1) \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{array} \right) \sim \begin{array}{c} \\ (1/3) \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2/3 & 0 \end{array} \right) \qquad x = t \begin{pmatrix} -1 \\ 2/3 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}.$

$\Rightarrow \mathcal{U}^\perp = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 2/3 \\ 1 \end{pmatrix} \right\} = \mathrm{span}\left\{ \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} \right\}$. $\quad \left\| \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} \right\| = \sqrt{(-3)^2 + 2^2 + 3^2} = \sqrt{22}$

$\Rightarrow \frac{1}{\sqrt{22}} \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$ is an orthonormal basis of $\mathcal{U}^\perp$

4a) Let $A, B \in \mathbb{R}^{n\times m}$, $\lambda \in \mathbb{R}$.

$$\left.\begin{array}{l} ev_v(A+B) = (A+B)v = Av + Bv = ev_v(A) + ev_v(B) \\ ev_v(\lambda A) = (\lambda A)v = \lambda(Av) = \lambda ev_v(A) \end{array}\right\} \Rightarrow \underline{\underline{ev_v \text{ is a linear map}}}$$

b) For $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, we have $Ae_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

Hence, $A \in \ker(ev_{e_1}) \Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow A$ is of the form $A = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}$

$\underline{\underline{\ker(ev_{e_1}) = \left\{ \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, c, d \in \mathbb{R} \right\}}}$

For any $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, let $A = \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix}$. Then $ev_{e_1}(A) = \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\Rightarrow y \in \operatorname{im}(ev_{e_1})$. Hence $\underline{\underline{\operatorname{im}(ev_{e_1}) = \mathbb{R}^2}}$.

c) If $v = 0$ then $ev_v(A) = 0$ for all $A \in \mathbb{R}^{n\times m}$, so $\underline{ev_v \text{ is not surjective}}$.

If $v \neq 0$: Let $y \in \mathbb{R}^n$, $y \neq 0$. Any vector $x \in \mathbb{R}^m$ can be written uniquely as $x = \lambda v + w$, where $w \in (\operatorname{span}\{v\})^\perp$.
Define a linear map $F_y : \mathbb{R}^m \to \mathbb{R}^n$ by $F(x) = \lambda y$ for x as above.
This map is clearly linear, and $F_y(v) = y$.

Setting $A_y = [F_y]$, we have $ev_v(A_y) = A_y v = F_y(v) = y \Rightarrow y \in \operatorname{im}(ev_v)$.

Hence, $\underline{\underline{ev_v \text{ is surjective}}}$.

d) If $v \neq 0$ then $\mathrm{im}(ev_v) = \mathbb{R}^n \Longrightarrow \dim(\mathrm{im}(ev_v)) = n$.

As $\dim(\mathbb{R}^{n \times m}) = nm$, it follows from the kernel-image theorem that

$$\dim(\ker(ev_v)) = \dim(\mathbb{R}^{n \times m}) - \dim(\mathrm{im}(ev_v)) = nm - n = \underline{\underline{(m-1)n}}$$