Linear algebra II, spring term 2017
Solutions to final exam, 25th July

1.a) $\det A = \begin{vmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 0 & 4/3 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot \frac{4}{3} \cdot (-1) \cdot 1 = \underline{\underline{-4}}$

b) $\det(A - \lambda I_4) = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 1 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)(-1-\lambda)\left((3-\lambda)(1-\lambda)+1\right) = (1-\lambda)(-1-\lambda)(\lambda^2 - 4\lambda + 4) = (1-\lambda)(-1-\lambda)(\lambda-2)^2$

Eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2}$

Algebraic multiplicities: $\underline{\mathrm{almu}_A(-1) = 1, \ \mathrm{almu}_A(1) = 1, \ \mathrm{almu}_A(2) = 2}$

Since $1 \leq \mathrm{gemu}_A(\lambda) \leq \mathrm{almu}_A(\lambda)$ for all eigenvalues $\lambda \in \mathbb{R}$

it follows that $\underline{\mathrm{gemu}_A(-1) = \mathrm{gemu}_A(1) = 1}$

$A - 2I_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \mathcal{E}_2(A) = \mathrm{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\} \Rightarrow \dim \mathcal{E}_2(A) = 1$, that is,

$\underline{\mathrm{gemu}_A(2) = 1}$

c) A is not diagonalisable, since $\operatorname{gemu}_A(2) < \operatorname{almu}_A(2)$

d) $U = \operatorname{span}\{Ae_1, Ae_2\}$, $Ae_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $Ae_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Since Ae_1 and Ae_2 are linearly independent and contained in $\operatorname{span}\{e_1, e_2\}$, it follows that $U = \operatorname{span}\{e_1, e_2\}$.

Hence (e_1, e_2) is an orthonormal basis of U.

2) The homogeneous equation $f''(t)+4f'(t)+4f(t)=0$ has characteristic polynomial $x^2+4x+4=(x+2)^2$.

$\Rightarrow$ $f_h(t)=c_1e^{-2t}+c_2te^{-2t}$ is the general solution of the homog. equation $(c_1,c_2\in\mathbb{R})$

Hence, $f(t)=f_h(t)+e^{-t}=c_1e^{-2t}+c_2te^{-2t}+e^{-t}$, $c_1,c_2\in\mathbb{R}$, is the general solution of the inhomogeneous equation.

3) $$B-1\cdot I_3=\begin{array}{c}1\\ \\ \\ \end{array}\begin{pmatrix}0&1&0\\0&0&2\\0&-1&3\end{pmatrix}\sim\begin{array}{c}\\ 1/2\\ \\ \end{array}\begin{pmatrix}0&1&0\\0&0&2\\0&0&3\end{pmatrix}\sim\begin{array}{c}\\ -3\\ \\ \end{array}\begin{pmatrix}0&1&0\\0&0&1\\0&0&3\end{pmatrix}\sim\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$\Rightarrow$ $\mathcal{E}_1(B)=\operatorname{span}\{e_1\}$. The vector $v_1=e_1$ is a basis of $\mathcal{E}_1(B)$

$$B-2I_3=\begin{pmatrix}-1&1&0\\0&-1&2\\0&-1&2\end{pmatrix}\begin{array}{c}\\ 1\quad -1\\ \\ \end{array}\sim\begin{array}{c}-1\\ -1\\ \\ \end{array}\begin{pmatrix}-1&0&2\\0&-1&2\\0&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&-2\\0&1&-2\\0&0&0\end{pmatrix}$$

$\Rightarrow$ $\mathcal{E}_2(B)=\operatorname{span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$

The vector $v_2=\begin{pmatrix}2\\2\\1\end{pmatrix}$ is a basis of $\mathcal{E}_2(B)$

$$B-3I_3=\begin{pmatrix}-2&1&0\\0&-2&2\\0&-1&1\end{pmatrix}\begin{array}{c}\\ \\ -2\quad 1\end{array}\sim\begin{array}{c}-1/2\\ \\ -1\end{array}\begin{pmatrix}-2&0&1\\0&0&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&0&-1/2\\0&0&0\\0&1&-1\end{pmatrix}\sim\begin{pmatrix}1&0&-1/2\\0&1&-1\\0&0&0\end{pmatrix}$$

$\Rightarrow$ $\mathcal{E}_3(B)=\operatorname{span}\left\{\begin{pmatrix}1/2\\1\\1\end{pmatrix}\right\}=\operatorname{span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}\right\}$

The vector $v_3=\begin{pmatrix}1\\2\\2\end{pmatrix}$ is a basis of $\mathcal{E}_3(B)$.

Together, the vectors $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\2\\1\end{pmatrix}$ and $v_3 = \begin{pmatrix}1\\2\\2\end{pmatrix}$ form an eigenbasis of $\mathbb{R}^3$ with respect to B.

b) $x(0) = \begin{pmatrix}1\\2\\2\end{pmatrix} = v_3 \implies Ax(0) = 3x(0)$, and

$$\underline{\underline{x(t) = A^t x(0) = 3^t x(0) = 3^t \begin{pmatrix}2\\1\\1\end{pmatrix}}}$$

4.a) Set $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $x^T A x = (x_1\ x_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_1 x_2 + x_2^2$
$= q(x)$

b) Set $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, where the numbers a_{ij} are those defining the form q.

Then $x^T A x = (x_1 \cdots x_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i,j=0}^{n} a_{ij} x_i x_j$

c) First: $x^T A^T x \in \mathbb{R} \Rightarrow (x^T A^T x)^T = x^T A^T x$.

Hence $x^T A^T x = (x^T A^T x)^T = x^T (A^T)^T (x^T)^T = x^T A x$,

and therefore $x^T A_S x = x^T \frac{1}{2}(A + A^T) x = \frac{1}{2}(x^T A x + x^T A^T x)$

$= \frac{1}{2}(x^T A x + x^T A x) = x^T A x$ for all $x \in \mathbb{R}^n$.

d) By (b), there exists a matrix A s.th. $q(x) = x^T A x$.

By (c), $q(x) = x^T A x = x^T A_S x$.

The matrix A_S is symmetric: $A_S^T = \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A^T + (A^T)^T)$

$= \frac{1}{2}(A^T + A) = A_S$.

e) Let A be a symmetric matrix such that $q(x) = x^T A x$

By the spectral theorem, there exists an orthonormal basis $\underline{v} = (v_1, \dots, v_n)$ of $\mathbb{R}^n$ such that $Av_i = \mu_i v_i$ for all $i = 1, \dots, n$ (where μ_i are the eigenvalues of A).

Now $q(\lambda_1 v_1 + \dots + \lambda_n v_n) = (\lambda_1 v_1 + \dots + \lambda_n v_n)^T A (\lambda_1 v_1 + \dots + \lambda_n v_n)$

$$= \sum_{j=1}^{n} (\lambda_1 v_1 + \dots + \lambda_n v_n)^T A (\lambda_j v_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i v_i)^T A (\lambda_j v_j) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j v_i^T A v_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j v_i^T (\mu_j v_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \mu_j v_i^T v_j$$

$$= \sum_{i=1}^{n} \lambda_i^2 \mu_i, \text{ because } \begin{cases} v_i^T v_i = 1, \\ v_i^T v_j = 0 \text{ if } i \neq j. \end{cases}$$

Moreover, $q(v_i) = v_i^T A v_i = v_i^T (\mu_i v_i) = \mu_i v_i^T v_i = \mu_i$,

so $\underline{\underline{q(\lambda_1 v_1 + \dots + \lambda_n v_n) = \sum_{i=1}^{n} \lambda_i^2 q(v_i)}}$