# Linear algebra I, autumn term 2018

## Solutions to Homework 5

1a) $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 7 & 7 & 8 & 0 \\ 1 & 2 & 1 & -1 \\ 7 & 7 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 7 & 1 & -7 \\ 0 & 2 & 0 & -2 \\ 0 & 7 & -1 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 7 & 1 & -7 \\ 0 & 7 & -1 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 7 & 1 & -7 \\ 0 & 7 & -1 & -7 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A).$

pivot el.

There are pivot elements in col. 1, 2 and 3 of $\operatorname{rref}(A)$, hence

$u_1 = Ae_1 = \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix}$, $u_2 = Ae_2 = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix}$, $u_3 = Ae_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix}$ form a basis of $\operatorname{im} A$.

Apply the Gram-Schmidt algorithm to obtain an orthonormal basis $\underline{f} = (f_1, f_2, f_3)$ from $(u_1, u_2, u_3)$:

$$f_1 = \frac{1}{\|u_1\|} u_1 = \frac{1}{10}\begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix}.$$

$$w_2 = u_2 - (u_2 \cdot f_1) f_1 = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix} - \frac{100}{10} \cdot \frac{1}{10} \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$w_3 = u_3 - (u_3 \cdot f_1) f_1 - (u_3 \cdot f_2) f_2 = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix} - \frac{100}{10} \cdot \frac{1}{10} \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix} - 0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$f_3 = \frac{1}{\|w_3\|} w_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

The vectors $f_1, f_2, f_3$ form an orthonormal basis of $\operatorname{im} A$.

b) To find an orthonormal basis $\underline{g} = (g_1, g_2, g_3, g_4)$ of $\mathbb{R}^4$ that extends $\underline{f}$, let $g_1 = f_1$, $g_2 = f_2$ and $g_3 = f_3$.
Then we need $g_4 \cdot f_1 = g_4 \cdot f_2 = g_4 \cdot f_3 = 0$, in other words,
$g_4 \in \operatorname{span}\{f_1, f_2, f_3\}^{\perp} = (\operatorname{im} F)^{\perp} = \ker A^T$, and $\|g_4\| = 1$.

$$A^T = \begin{pmatrix} 1 & 7 & 1 & 7 \\ 0 & 7 & 2 & 7 \\ 1 & 8 & 1 & 6 \\ 1 & 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & 1 & 7 \\ 0 & 7 & 2 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & -7 & -2 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 7 & 2 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 14 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A^T)$$

Hence $A^T x = 0 \iff \begin{cases} x_1 + 7x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 + 7x_4 = 0 \end{cases}$ Set $x_4 = t$. $x = t\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

$(\operatorname{im} F)^{\perp} = \operatorname{span}\left\{\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix}\right\}$.

So $g_4 \in \operatorname{span}\left\{\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix}\right\}$, $\|g_4\| = 1$.

Define $g_4 = \dfrac{1}{\left\|\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix}\right\|}\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix} = \dfrac{1}{10}\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix}$.

Then $g_4 \in \operatorname{span}\left\{\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix}\right\} = (\operatorname{im} F)^{\perp} \implies g_1 \cdot g_4 = g_2 \cdot g_4 = g_3 \cdot g_4 = 0$, $\|g_4\| = 1$.

Hence $\underline{g} = (g_1, g_2, g_3, g_4)$ is an orthonormal basis of $\mathbb{R}^4$, extending $\underline{f}$.

$$\left( g_1 = \frac{1}{10}\begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix},\ g_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ g_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},\ g_4 = \frac{1}{10}\begin{pmatrix} -7 \\ 1 \\ -7 \\ 1 \end{pmatrix} \right)$$

2) Define a linear map $F: \mathbb{R}^3 \to \mathbb{R}^4$ by $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{pmatrix}$

where $f(t) = a + b\cos\frac{\pi t}{2} + c\sin\frac{\pi t}{2}$.

Then the least square approximation of the points $(0,y_0), (1,y_1), (2,y_2), (3,y_3)$

is given by $f(t) = \alpha + \beta\cos\frac{\pi t}{2} + \gamma\sin\frac{\pi t}{2}$, where $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = x$ is

a solution to the equation $A^TAx = A^Tb$, for $A = [F]$ and $b = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

$$F(e_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad F(e_2) = \begin{pmatrix} \cos(0) \\ \cos\frac{\pi}{2} \\ \cos\pi \\ \cos\frac{3\pi}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad F(e_3) = \begin{pmatrix} \sin(0) \\ \sin\frac{\pi}{2} \\ \sin\pi \\ \sin\frac{3\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow A = [F] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_0 + y_1 + y_2 + y_3 \\ y_0 - y_2 \\ y_1 - y_3 \end{pmatrix}$$

Hence, $A^TAx = A^Tb$

$$\Longleftrightarrow \quad x = \begin{pmatrix} (y_0+y_1+y_2+y_3)/4 \\ (y_0-y_2)/2 \\ (y_1-y_3)/2 \end{pmatrix}.$$

The least-square approximation of the points is

$$\underline{\underline{f(t) = \frac{y_0+y_1+y_2+y_3}{4} + \frac{y_0-y_2}{2}\cos\frac{\pi t}{2} + \frac{y_1-y_3}{2}\sin\frac{\pi t}{2}}}.$$

3) Assume that $\underline{u} = (u_1, \dots, u_r)$ and $\underline{v} = (v_1, \dots, v_s)$ are bases of $U$ and $U^\perp$, respectively.

By results from the lectures, we know that every $x \in \mathbb{R}^n$ can be written as $x = x_\parallel + x_\perp$ for some $x_\parallel \in U$, $x_\perp \in U^\perp$.
Hence, $x_\parallel \in \operatorname{span} \underline{u} \subset \operatorname{span}(\underline{u} \cup \underline{v})$ and $x_\perp \in \operatorname{span} \underline{v} \subset \operatorname{span}(\underline{u} \cup \underline{v})$

$$\Longrightarrow x = x_\parallel + x_\perp \in \operatorname{span}(\underline{u} \cup \underline{v}).$$

To show that $\underline{u} \cup \underline{v}$ is linearly independent, assume that $\lambda_1 u_1 + \dots + \lambda_r u_r + \mu_1 v_1 + \dots + \mu_s v_s = 0$, where $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s \in \mathbb{R}$.

Then $\underbrace{\lambda_1 u_1 + \dots + \lambda_r u_r}_{\in \operatorname{span} \underline{u}} = \underbrace{-\mu_1 v_1 - \mu_2 v_2 - \dots - \mu_s v_s}_{\in \operatorname{span} \underline{v}} \in (\operatorname{span} \underline{u}) \cap (\operatorname{span} \underline{v}) = U \cap U^\perp = \{0\}$

Hence $\lambda_1 u_1 + \dots + \lambda_r u_r = 0 \Longrightarrow \lambda_1 = \dots = \lambda_r = 0$,
and $\mu_1 v_1 + \dots + \mu_s v_s = 0 \Longrightarrow \mu_1 = \dots = \mu_s = 0$
since $\underline{u}$ and $\underline{v}$ each is linearly independent.

Thus, $\underline{u} \cup \underline{v} = (u_1, \dots, u_r, v_1, \dots, v_s)$ is linearly independent.

<u>This proves that $\underline{u} \cup \underline{v}$ is a basis of $\mathbb{R}^n$</u>.