Linear algebra I, autumn term 2018
Solutions to Homework 4

1) The problems (a) and (b) may be solved simultaneously.
Consider the augmented matrix of the equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = x$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 3 & 1 & 3+a \\ 1 & 0 & 2 & 1 & 3+a \\ 1 & 0 & 2 & 0 & 2+2a \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 0 & -2 & 0 & -a \\ 2 & 1 & 3 & 1 & 3+a \\ 1 & 0 & 2 & 1 & 3+a \\ 1 & 0 & 2 & 0 & 2+2a \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 0 & -2 & 0 & -a \\ 0 & 1 & -1 & 1 & 3-a \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2+a \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & a \\ 0 & 1 & -1 & 0 & -a \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2+a \end{array}\right).$$

a) From the above computation, it follows that the eqation

$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0$ has augmented matrix $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

Since the row-reduced echelon form has
pivot elements in columns 1, 2 and 4, it follows that $\underline{b} = (u_1, u_2, u_4)$
is a basis of U.
Moreover, the augmented matrix of the equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_4 u_4 = u_3$

is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, so $u_3 = 2u_1 - u_2 + 0 \cdot u_4$.

Hence: $\underline{b} = (u_1, u_2, u_4)$ is a basis of U, and $[u_1]_{\underline{b}} = e_1$, $[u_2]_{\underline{b}} = e_2$, $[u_3]_{\underline{b}} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$,
$[u_4]_{\underline{b}} = e_3$

b) By the calculation in the beginning, the equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = x$ is soluble if and only if $a = -2$.
Hence, $\underline{x \in U \iff a = -2.}$

In this case, the eq. $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_4 u_4 = x$ has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so $x = -2u_1 + 2u_2 + 3u_4$ and thus

$$\underline{[x]_{\underline{b}} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}.}$$

2.a) $$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the above computation follows that the equation $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$ has unique solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so v_1, v_2, v_3 are linearly independent. Since $\dim \mathbb{R}^3 = 3$, it follows that $\underline{v} = (v_1, v_2, v_3)$ is a basis of $\mathbb{R}^3$.

b) $F(v_1) = Av_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = v_2 \implies [F(v_1)]_{\underline{v}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$F(v_2) = Av_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = v_3 \implies [F(v_2)]_{\underline{v}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$F(v_3) = Av_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. The eq. $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ has augm. matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$\implies F(v_3) = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 2v_1 - v_2 + v_3 \implies [F(v_3)]_{\underline{v}} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Hence, $\underline{[F]_{\underline{v}} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.}$

3. a) · Let $x, y \in U^\perp$, i.e., $x \cdot u = 0$ and $y \cdot u = 0$ for all $u \in U$.
Then $(x+y) \cdot u = x \cdot u + y \cdot u = 0 + 0 = 0$ for all $u \in U$, so $\underline{x+y \in U^\perp}$.

· Let $x \in U^\perp$, $\lambda \in \mathbb{R}$. Then, for all $u \in U$: $(\lambda x) \cdot u = \lambda (x \cdot u) = 0$
$\Longrightarrow$ $\underline{\lambda x \in U^\perp}$.

· For all $u \in U$, the equation $0 \cdot u = 0$ holds, so $\underline{0 \in U^\perp}$.

Hence, $U^\perp$ is a subspace of $\mathbb{R}^n$.

b) "$\Rightarrow$": Assume that $x \in U^\perp$. Then $x \cdot u = 0$ for all $u \in U$ so, in particular, $\underline{x \cdot u_1 = \dots = x \cdot u_r = 0}$.

"$\Leftarrow$": Assume that $x \cdot u_1 = \dots = x \cdot u_r = 0$, and let $w \in U$.
Since $U = \text{span}\{u_1, \dots, u_r\}$, there exist $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that $w = \lambda_1 u_1 + \dots + \lambda_r u_r$.
Hence
$$x \cdot w = x \cdot (\lambda_1 u_1 + \dots + \lambda_r u_r) = x \cdot (\lambda_1 u_1) + \dots + x \cdot (\lambda_r u_r)$$
$$= \lambda_1 (x \cdot u_1) + \dots + \lambda_r (x \cdot u_r) = 0 + \dots + 0 = 0$$
This shows that $x \cdot w = 0$ for all $w \in U$, that is, $\underline{x \in U^\perp}$.