Linear Algebra I, autumn term 2018
Solutions to Homework 3

pivot el.

1) $A = \begin{array}{c} (-3) \\ \end{array} \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} \sim \begin{array}{c} (1/2) \\ \end{array} \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix} = \mathrm{rref}(A)$

Hence, $Ax = 0 \iff x = t\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ for some $t \in \mathbb{R}$.

So the vector $v = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ forms a basis of $\ker A$.

Since $\mathrm{rref}(A)$ has a pivot element only in the first column, it follows that $Ae_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ forms a basis of $\mathrm{im}\, A$.

$B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \mathrm{rref}(B)$. $Bx = 0 \iff x = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ for some $s, t \in \mathbb{R}$.

$\Rightarrow \left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right)$ is a basis of $\ker B$.

Clearly, $\mathrm{im}\, B = \mathbb{R}$, so the number 1 forms a basis of $\mathrm{im}\, B$.

pivot el.

$C = \begin{array}{c} (-1) \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(C)$. $Cx = 0 \iff x = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ for some $s, t \in \mathbb{R}$

$\Rightarrow \left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$ is a basis of $\ker C$.

Moreover, $Ce_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ forms a basis of $\mathrm{im}\, C$.

$P_u : \mathbb{R}^2 \to \mathbb{R}^2,\ x \longmapsto \frac{u \cdot x}{u \cdot u} u$, where $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2$.

$P_u(x) = 0 \iff u \cdot x = 0 \iff x_1 - x_2 = 0 \iff x_1 = x_2$.

Hence $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ forms a basis of $\ker P_u$.

$$\left.\begin{array}{l} \forall x \in \mathbb{R}^2 : P_u(x) = \frac{u \cdot x}{u \cdot u} u \in \mathrm{span}\{u\} \Rightarrow \mathrm{im}\, P_u \subset \mathrm{span}\{u\} \\ u = P_u(u) \in \mathrm{im}\, P_u \Rightarrow \mathrm{span}\{u\} \subset \mathrm{im}\, P_u \end{array}\right\} \Rightarrow \mathrm{im}\, P_u = \mathrm{span}\{u\}$$

Hence, u forms a basis of $\mathrm{im}\, P_u$.

$\mathrm{rot}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2,\ x \longmapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} x$

($\|\mathrm{rot}_\alpha(x)\| = \|x\|$ for all $x \in \mathbb{R}^2$, so $\mathrm{rot}_\alpha(x) = 0 \iff x = 0$.
Hence, $\ker(\mathrm{rot}_\alpha) = \{0\}$, so the empty set $\emptyset$ is a basis of $\ker(\mathrm{rot}_\alpha)$.)

$\forall y \in \mathbb{R}^2 : y = \mathrm{rot}_\alpha(\mathrm{rot}_{-\alpha}(y)) \in \mathrm{im}(\mathrm{rot}_\alpha)$. Hence, $\mathrm{im}(\mathrm{rot}_\alpha) = \mathbb{R}^2$, and $\underline{e} = (e_1, e_2)$ is a basis of $\mathrm{im}(\mathrm{rot}_\alpha)$.

$F_{v,w} : \mathbb{R}^2 \to \mathbb{R}^2,\ x \mapsto x_1 v + x_2 w$. $\mathrm{im}\, F_{v,w} = \{x_1 v + x_2 w \in \mathbb{R}^2 \mid x_1, x_2 \in \mathbb{R}\} = \mathrm{span}\{v, w\}$.

There are two cases:

Case 1: The vectors v, w are linearly independent:
Then (v, w) is a basis of $\mathrm{im}(F_{v,w})$.

Case 2: The vectors v, w are linearly dependent:
Since $v \neq 0$, this means that $w \in \mathrm{span}\{v\} \Rightarrow \mathrm{im}(F_{v,w}) = \mathrm{span}\{v\}$.
Hence, in this case, v forms a basis of $\mathrm{im}(F_{v,w})$.

2) $\mathcal{U} \subset \mathbb{R}^3$ is not a subspace:

Let $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$. Then $0 \leq 0 \leq 1 \Rightarrow u \in \mathcal{U}$.
But $(-1)u = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, and $0 \leq 0 \not\leq -1$, so $(-1)u \notin \mathcal{U}$.

$V \subset \mathbb{R}^2$ is a subspace:

$$V = \left\{ \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x,y \in \mathbb{R} \right\}$$

$$v \in V \iff \exists x,y \in \mathbb{R}: \ v = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix} = x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Hence $V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$, which is a subspace of $\mathbb{R}^2$.

$W_{A,B} \subset \mathbb{R}^n$ is a subspace, for all $A,B \in \mathbb{R}^{n\times n}$:

$$x \in W_{A,B} \iff Ax = Bx \iff (A-B)x = 0 \iff x \in \ker(A-B)$$

So $W_{A,B} = \ker(A-B)$, which is a subspace of $\mathbb{R}^n$.

3a) $W = \operatorname{span}\{v_1, v_2, v_3, v_4\}$. Check if v_1, v_2, v_3, v_4 are linearly independent:

Assume that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$. This gives a linear system with

augmented matrix $\left(\begin{array}{cccc|c} | & | & | & | & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 \\ | & | & | & | & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 5 & 0 \\ 1 & 2 & 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$. The row-reduced echelon form of this matrix has pivot elements in the first and third columns $\Longrightarrow$ $\underline{v} = (v_1, v_3)$ is a basis of W.

b) $\underline{v_1 = 1\cdot v_1 + 0\cdot v_3}$; $\underline{v_2 = 2v_1 = 2\cdot v_1 + 0\cdot v_3}$; $\underline{v_3 = 0\cdot v_1 + 1\cdot v_3}$

The equation $\lambda_1 v_1 + \lambda_3 v_3 = v_4$ has augmented matrix $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 2 & -1 \end{array}\right)$.

By the calculations in (a), $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 5 \\ 1 & 2 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$ (read columns 1, 2 and 4 in the augm. matrix).

Hence, $\lambda_1 = 3$ and $\lambda_3 = -2$, so $\underline{v_4 = 3v_1 - 2v_3}$

c) $\dim W = 2$, since $\underline{v} = (v_1, v_3)$ is a basis of W.

4a) $\underline{U+V \subset \mathbb{R}^n \text{ is a subspace:}}$

- $0 = 0 + 0 \in U + V$ ($0 \in U$ and $0 \in V$ because U and V are subspaces)

- If $u_1 + v_1 \in U+V$ and $u_2 + v_2 \in U+V$ then $(u_1+v_1) + (u_2+v_2) = (u_1+u_2) + (v_1+v_2) \in U+V$
  ($u_1+u_2 \in U$, $v_1+v_2 \in V$ since U and V are subspaces of $\mathbb{R}^n$)
- If $u+v \in U+V$ and $\lambda \in \mathbb{R}$, then $\lambda(u+v) = \lambda u + \lambda v \in U+V$
  ($\lambda u \in U$ and $\lambda v \in V$ since U and V are subspaces)

$\underline{U \cap V \text{ is a subspace:}}$

- $0 \in U$, $0 \in V \Rightarrow 0 \in U \cap V$.

- If $x, y \in U \cap V$. Then $x, y \in U \Rightarrow x+y \in U$, and $x, y \in V \Rightarrow x+y \in V$.
  Hence $x+y \in U \cap V$.
- If $x \in U \cap V$, $\lambda \in \mathbb{R}$: Then $x \in U \Rightarrow \lambda x \in U$, and $x \in V \Rightarrow \lambda x \in V$.
  Hence $\lambda x \in U \cap V$.

<u>$U \cup V$ is not necessarily a subspace:</u>

Let $U = \operatorname{span}\{e_1\} \subset \mathbb{R}^2$ and $V = \operatorname{span}\{e_2\} \subset \mathbb{R}^2$.

Then $e_1 \in U \subset U \cup V$ and $e_2 \in V \subset U \cup V$, but
$e_1 + e_2 = \begin{pmatrix}1\\1\end{pmatrix} \notin U$ and $e_1 + e_2 \notin V$, so $e_1 + e_2 \notin U \cup V$.
Hence $U \cup V$ is not closed under sums.

5) Assume that $U, V, W \subset \mathbb{R}^n$ are subspaces, $U \subset W$ and $V \subset W$.

Let $x \in U + V$. Then $x = u + v$ for some $u \in U$, $v \in V$.
Now $u \in U \subset W$, $v \in V \subset W \implies u + v \in W$, that is, $x \in W$.

Hence, every element x of $U+V$ belongs to W, that is, <u>$U + V \subset W$</u>.