<u>Linear algebra I, autumn term 2018</u>

<u>Solutions to Homework 2</u>

1a)

$$\begin{array}{c} \textcircled{-1}\ \textcircled{1} \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ -2 & 1 & 6 & 0 & -6 & -12 \\ 0 & 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 2 & 1 & -3 & 0 & 8 & 7 \end{pmatrix} \sim \begin{array}{c} \\ 2\ \ -1 \\ \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{array}{c} \\ \\ -\frac{1}{2} \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 0 & -6 & 0 & 0 & 10 \\ 0 & 0 & 6 & 1 & 3 & -9 \\ 0 & 0 & -3 & 0 & 0 & 5 \end{pmatrix} \sim \begin{array}{c} \\ \\ 1\ \ -2 \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 0 & 3 & 0 & 0 & -5 \\ 0 & 0 & 6 & 1 & 3 & -9 \\ 0 & 0 & -3 & 0 & 0 & 5 \end{pmatrix}$$

$$\sim \begin{array}{c} \\ \\ 1\ \ -1 \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & -3 & 0 & 7 & 7 \\ 0 & 1 & 3 & 0 & 1 & -5 \\ 0 & 0 & 3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{array}{c} 1/2 \\ \\ 1/3 \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & 0 & 0 & 7 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 7/2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(B)$$

b) The augmented matrix of the equation $Bx=0$ is

$$(B|0) \overset{(\text{by } 1.a)}{\sim} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 7/2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -5/3 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Setting $x_5 = s$ and $x_6 = t$ we have

$$\begin{cases} x_1 = -\frac{7}{2}s - t \\ x_2 = -s \\ x_3 = \frac{5}{3}t \\ x_4 = -3s - t \\ x_5 = s \\ x_6 = t \end{cases} \quad \text{that is,} \quad x = \begin{pmatrix} -\frac{7}{2}s - t \\ -s \\ \frac{5}{3}t \\ -3s - t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -\frac{7}{2} \\ -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 5/3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad s,t \in \mathbb{R}$$

2) (i) The map F is linear:

For all $u, v \in \mathbb{R}^3$:

$$F(u+v) = F\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} = \begin{pmatrix} (u_1+v_1) - 3(u_2+v_2) \\ 0 \\ 2(u_1+v_1) + \frac{1}{2}(u_2+v_2) + (u_3+v_3) \end{pmatrix}$$

$$= \begin{pmatrix} (u_1-3u_2) + (v_1-3v_2) \\ 0 \\ (2u_1+\frac{1}{2}u_2+u_3) + (2v_1+\frac{1}{2}v_2+v_3) \end{pmatrix} = \begin{pmatrix} u_1-3u_2 \\ 0 \\ 2u_1+\frac{1}{2}u_2+u_3 \end{pmatrix} + \begin{pmatrix} v_1-3v_2 \\ 0 \\ 2v_1+\frac{1}{2}v_2+v_3 \end{pmatrix} = F(u)+F(v)$$

For all $u \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$:

$$F(\lambda u) = F\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 - 3\lambda u_2 \\ 0 \\ 2\lambda u_1 + \frac{1}{2}\lambda u_2 + \lambda u_3 \end{pmatrix} = \begin{pmatrix} \lambda(u_1-3u_2) \\ 0 \\ \lambda(u_1+\frac{1}{2}u_2+u_3) \end{pmatrix} = \lambda\begin{pmatrix} u_1-3u_2 \\ 0 \\ u_1+\frac{1}{2}u_2+u_3 \end{pmatrix}$$

$= \lambda F(u)$. Hence, F is linear.

As $F(e_1) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $F(e_2) = \begin{pmatrix} -3 \\ 0 \\ 1/2 \end{pmatrix}$ and $F(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we have

$$[F] = \begin{pmatrix} | & | & | \\ F(e_1) & F(e_2) & F(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 1/2 & 1 \end{pmatrix}.$$

(ii) The map G is not linear:

Setting $u=v=0$, we have

$$G(u+v) = G(0+0) = G(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\neq$$
$$G(u)+G(v) = G(0)+G(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

so the first condition in the definition of a linear map is not satisfied.

(iii) The map α is not linear:

Let $x = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\lambda = -1$.

Then $\alpha(\lambda x) = \alpha\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$$\neq$$
$$\lambda\alpha(x) = (-1)\alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Hence, the second condition in the definition of a linear map is not satisfied.

(iv) The map $\beta = \beta_{(u,v,w)}$ is linear:

For all $x, y \in \mathbb{R}^3$,

$$\beta(x+y) = \beta\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = (x_1+y_1)u + (x_2+y_2)v + (x_3+y_3)w$$

$$= x_1u + y_1u + x_2v + y_2v + x_3w + y_3w = (x_1u + x_2v + x_3w) + (y_1u + y_2v + y_3w)$$

$$= \beta(x) + \beta(y).$$

For all $x \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$,

$$\beta(\lambda x) = \beta\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} = (\lambda x_1)u + (\lambda x_2)v + (\lambda x_3)w = \lambda\left(x_1u + x_2v + x_3w\right) = \lambda F(u).$$

Hence β is linear.

Moreover, $\beta(e_1) = u$, $\beta(e_2) = v$, $\beta(e_3) = w$, so

$$[\beta] = [\beta_{(u,v,w)}] = \begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}$$

3) We assume first that such a map F exists, and try to determine its matrix. To do this, we must compute $F(e_1)$ and $F(e_2)$.

Set $u = \begin{pmatrix}1\\2\end{pmatrix}$ and $v = \begin{pmatrix}3\\5\end{pmatrix}$

Find $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 u + \lambda_2 v = e_1$:

$$\lambda_1 \begin{pmatrix}1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \iff \begin{pmatrix}\lambda_1 + 3\lambda_2\\ 2\lambda_1 + 5\lambda_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

Augmented matrix: $\left(\begin{array}{cc|c}1 & 3 & 1\\ 2 & 5 & 0\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 3 & 1\\ 0 & -1 & -2\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & -5\\ 0 & -1 & -2\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & -5\\ 0 & 1 & 2\end{array}\right)$

that is, $\lambda_1 = -5$ and $\lambda_2 = 2$,

so $e_1 = -5u + 2v$.

Similarly, $\mu_1 u + \mu_2 v = e_2 \iff \begin{pmatrix}\mu_1 + 3\mu_2\\ 2\mu_1 + 5\mu_2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, which has

augmented matrix $\left(\begin{array}{cc|c}1 & 3 & 0\\ 2 & 5 & 1\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 3 & 0\\ 0 & -1 & 1\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & 3\\ 0 & -1 & 1\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & 3\\ 0 & 1 & -1\end{array}\right)$

$\Rightarrow \mu_1 = 3$, $\mu_2 = -1$, and so $e_2 = 3u - v$.

Now, $F(e_1) = F(-5u + 2v) = (-5)F(u) + 2F(v) = (-5)\begin{pmatrix}1\\2\\3\end{pmatrix} + 2\begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}-1\\-6\\-17\end{pmatrix}$

and $F(e_2) = F(3u - v) = 3F(u) - F(v) = 3\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\4\\10\end{pmatrix}$.

Hence, if a map F with the required properties exist, its matrix must be $[F] = A = \begin{pmatrix}-1 & 1\\ -6 & 4\\ -17 & 10\end{pmatrix}$.

On the other hand, the linear map $F_A: \mathbb{R}^2 \to \mathbb{R}^3$, $F_A(x) = Ax$

satisfies $F_A\begin{pmatrix}1\\2\end{pmatrix} = A\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-1 & 1\\-6 & 4\\-17 & 10\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and

$$F_A\begin{pmatrix}3\\5\end{pmatrix} = A\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-1 & 1\\-6 & 4\\-17 & 10\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}.$$

Hence, F_A is the unique linear map $\mathbb{R}^2 \to \mathbb{R}^3$ satisfying the required conditions.

The matrix of F_A is $[F_A] = A = \begin{pmatrix}-1 & 1\\-6 & 4\\-17 & 10\end{pmatrix}$.

4) $A = \begin{pmatrix}-2 & 6\\1 & -3\end{pmatrix}$. $Ax = 0 \iff \begin{cases}-2x_1 + 6x_2 = 0\\ x_1 - 3x_2 = 0 \ (2)\end{cases} \iff x - 3x_2 = 0$

$\iff x = \begin{pmatrix}3t\\t\end{pmatrix}$, where $t \in \mathbb{R}^3$.

So $V = \left\{\begin{pmatrix}3t\\t\end{pmatrix} \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R}\right\}$

Define $f: \mathbb{R} \to V$ by $f(t) = \begin{pmatrix}3t\\t\end{pmatrix}$.

Then $f(t) = \begin{pmatrix}3t\\t\end{pmatrix} \in V$ for all $t \in \mathbb{R}$.

Conversely, define $g: V \to \mathbb{R}$ by $g\begin{pmatrix}v_1\\v_2\end{pmatrix} = v_2$ for all $v = \begin{pmatrix}v_1\\v_2\end{pmatrix} \in V$.

Then, $\forall t \in \mathbb{R}$: $(g \circ f)(t) = g(f(t)) = g\begin{pmatrix}3t\\t\end{pmatrix} = t$, and

$\forall v = \begin{pmatrix}v_1\\v_2\end{pmatrix} \in V$: $(f \circ g)\begin{pmatrix}v_1\\v_2\end{pmatrix} = f\left(g\begin{pmatrix}v_1\\v_2\end{pmatrix}\right) = f(v_2) = \begin{pmatrix}3v_2\\v_2\end{pmatrix}$.

Since $v \in V = \left\{ \begin{pmatrix} 3t \\ t \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R} \right\}$, we know that $v_1 = 3v_2$.

Hence, $(f \circ g)(v) = \begin{pmatrix} 3v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v$.

This means that $f : \mathbb{R}^2 \to \mathbb{R}^3$ is invertible, and $f^{-1} = g$.

5) We shall show that $\begin{cases} (X \cup Y) \cap Z \subset (X \cap Z) \cup (Y \cap Z) \quad (1) \text{ and} \\ (X \cap Z) \cup (Y \cap Z) \subset (X \cup Y) \cap Z \quad (2) \end{cases}$

and thus that $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$.

First, (1): Let $a \in (X \cup Y) \cap Z$.
Then $a \in X \cup Y$ and $a \in Z$.

This means that $a \in X$ or $a \in Y$.
If $a \in X$: Then $a \in X \cap Z$ and thus $a \in (X \cap Z) \cup (Y \cap Z)$.
If $a \in Y$: Then $a \in Y \cap Z$ and thus $a \in (X \cap Z) \cup (Y \cap Z)$.
In either case, $a \in (X \cap Z) \cup (Y \cap Z)$.
Hence $(X \cup Y) \cap Z \subset (X \cap Z) \cup (Y \cap Z)$ holds.

Inclusion (2): Let $b \in (X \cap Z) \cup (Y \cap Z)$.
Then either $b \in X \cap Z$ or $b \in Y \cap Z$.

If $b \in X \cap Z$: $\left.\begin{array}{l} b \in X \Rightarrow b \in X \cup Y \\ b \in Z \end{array}\right\} \Longrightarrow b \in (X \cup Y) \cap Z$.

If $b \in Y \cap Z$: $\left.\begin{array}{l} b \in Y \Rightarrow b \in X \cup Y \\ b \in Z \end{array}\right\} \Rightarrow b \in (X \cup Y) \cap Z$.

So, in either case,
$$b \in (X \cup Y) \cap Z.$$
Hence, $(X \cap Z) \cup (Y \cap Z) \subset (X \cup Y) \cap Z$ holds.

Since both inclusions (1) and (2) hold, it follows that

$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$, q.e.d.