Linear algebra I, autumn term 2018
Solutions to Homework 1

1.a)
$$\begin{cases} x+2y+3z=1 \\ 4x+5y+6z=4 \\ 7x+8y+10z=1 \end{cases} \iff \begin{cases} x+2y+3z=1 \\ -3y-6z=0 \\ -6y-11z=-6 \end{cases} \iff \begin{cases} x+2y+3z=1 \\ -3y-6z=0 \\ z=-6 \end{cases}$$

$$\iff \begin{cases} x+2y+3z=1 \\ y+2z=0 \\ z=-6 \end{cases} \iff \begin{cases} x+2y=19 \\ y=12 \\ z=-6 \end{cases} \iff \underbrace{\begin{cases} x=-5 \\ y=12 \\ z=-6 \end{cases}}_{\text{r.r.e.f.}}$$

The system has the unique solution $\begin{cases} x=-5 \\ y=12 \\ z=-6 \end{cases}$

1.b)
$$\begin{cases} x+2y+3z=1 \\ 2x+3y+4z=1 \\ 3x+4y+6z=1 \\ x+y+2z=1 \end{cases} \iff \begin{cases} x+2y+3z=1 \\ -y-2z=-1 \\ -2y-3z=-2 \\ -y-z=0 \end{cases}$$

$$\begin{cases} x+2y+3z=1 \\ -y-2z=-1 \\ z=0 \\ z=1 \end{cases} \iff \underbrace{\begin{cases} x+2y+3z=1 \\ -y-2z=-1 \\ z=0 \\ 0=1 \end{cases}}_{\circledast}$$

This system has no solutions.

2) The row-reduced echelon form of the system (a) is

$$\begin{cases} x \quad\quad = -5 \\ \quad y \quad = 12 \\ \quad\quad z = -6 \end{cases}$$

The system (b): Continuing from ⊛, we have

$$\begin{cases} x+2y+3z=1 \\ -y-2z=-1 \\ z=0 \\ 0=1 \end{cases} \begin{matrix} \textcircled{-3} \\ \textcircled{2} \end{matrix} \iff \begin{cases} x+2y \quad = 1 \\ -y \quad = -1 \\ z=0 \\ 0=1 \end{cases} \textcircled{2}$$

$$\iff \begin{cases} x \quad = -1 \\ -y = -1 \\ z=0 \\ 0=1 \end{cases} \textcircled{-1} \iff \begin{cases} x \quad = -1 \\ y = 1 \\ z=0 \\ 0=1 \end{cases}$$

This is the row-reduced echelon form of the system (b)!

Remark: In (b), any inconsistent system on row-reduced echelon form will do, for example the system $\boxed{0=1}$ (which has one equation and zero variables).

3) The system (c) is on row-reduced echelon form, (a) and (b) are not.

(b): $2x+3y+4z+5w=6 \iff x+\frac{3}{2}y+2z+\frac{5}{2}w=3$

This is on r.r.e.f.

Solutions: Set $y=r$, $z=s$, $w=t$:

$$\begin{cases} x=3-\frac{3}{2}r-2s-\frac{5}{2}t \\ y=r \\ z=s \\ w=t \end{cases} \quad r,s,t\in\mathbb{R}$$

(a): $\begin{cases} x_1+x_2+x_3 = 0 \\ x_2 + x_4 = 0 \\ x_5 = 0 \end{cases} \iff \begin{cases} x_1 + x_3 - x_4 = 0 \\ x_2 + x_4 = 0 \\ x_5 = 0 \end{cases}$ This is on r.r.e.f.

Solutions: Set $x_3=s$, $x_4=t$:

$$\begin{cases} x_1=-s+t \\ x_2=-t \\ x_3=s \\ x_4=t \\ x_5=0 \end{cases} \quad s,t\in\mathbb{R}$$

(c) Unique solution $\begin{cases} x=-1 \\ y=0 \\ z=1 \end{cases}$

4) Let s denote the total amount of steel needed for the production of one unit of P, and e the total amount of electricity required for the same production.

Then we know that $s = 500 + \frac{1}{10}e$, and $e = 1000 + 5s$, hence, we get a linear system as follows:

$$\begin{cases} s - \frac{1}{10}e = 500 \\ -5s + e = 1000 \end{cases} \iff \begin{cases} s - \frac{1}{10}e = 500 \\ \frac{1}{2}e = 3500 \end{cases}$$

$$\iff \begin{cases} s - \frac{1}{10}e = 500 \\ e = 7000 \end{cases} \iff \begin{cases} s = 1200 \\ e = 7000 \end{cases}$$

<u>In other words, 1200 units of steel, and 7000 units of electricity is needed for the production of one unit of P.</u>