# Linear algebra I, autumn term 2018
## Solutions to midterm exam, 26th November 2018

1.a) The matrix-vector form of the system is $Ax = b$,

where $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ -1 & 0 & -4 & 1 \\ 1 & 1 & 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$.

b) The augmented matrix of the system is

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ -1 & 0 & -4 & 1 & 2 \\ 1 & 1 & 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 0 & 2 & -2 & 4 & 4 \\ 0 & -1 & 1 & -2 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & -1 & 1 & -2 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 4 & -1 & -2 \\ 0 & 1 & -1 & 2 & 2 \\ \underbrace{0 \quad 0 \quad 0 \quad 0}_{\mathrm{rref}(A)} & & & & 0 \end{array}\right) \qquad \begin{cases} x_1 + 4x_3 - x_4 = -2 \\ x_2 - x_3 + 2x_4 = 2 \end{cases}$$

Set $x_3 = s$, $x_4 = t$:

The solutions are: $\begin{cases} x_1 = -2 - 4s + t \\ x_2 = 2 + s - 2t \\ x_3 = s \\ x_4 = t \end{cases}$ where $s, t \in \mathbb{R}$.

c) By the calculation in (b), $A \sim \begin{pmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A)$.

The matrix $\mathrm{rref}(A)$ has two pivot elements, and hence $\mathrm{rank}(A) = 2$.

2) • Set $\lambda=-1$ and $x=e_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$.

Then $F(\lambda x)=F\begin{pmatrix}0\\-1\\0\end{pmatrix}=\begin{pmatrix}0\\(-1)^2\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, but

$\lambda F(x)=(-1)F\begin{pmatrix}0\\1\\0\end{pmatrix}=-\begin{pmatrix}0\\1^2\\0\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}$, so $F(\lambda x)\neq\lambda F(x)$ and thus <u>$F$ is not a linear map</u>.

• Let $x,y\in\mathbb{R}^2$. Then $G(x+y)=u\cdot(x+y)=u\cdot x+u\cdot y=G(x)+G(y)$.

Let $x\in\mathbb{R}^2$, $\lambda\in\mathbb{R}$. Then $G(\lambda x)=u\cdot(\lambda x)=\lambda(u\cdot x)=\lambda G(x)$.

Hence, <u>$G$ is a linear map</u>.

• Let $a,b,c\in\mathbb{R}$, and $f(t)=a+bt+ct^2$.

Then $f(0)=a$,
$f'(t)=b+2ct \Rightarrow f'(0)=b$,
$f''(t)=2c \Rightarrow f''(0)=2c$,

so $H\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\b\\2c\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$.

$\Rightarrow$ <u>$H$ is a linear map</u>, and <u>$[H]=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$</u>.

• The matrix of $G$: Let $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$. Then $G(e_1)=u\cdot e_1=u_1$ and
$G(e_2)=u\cdot e_2=u_2$

$\Longrightarrow$ <u>$[G]=u^T=(u_1\ \ u_2)$</u>.

3) Let $G = \mathrm{rot}_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$;
so that $G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$.

Then $G^2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G\left(G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = G\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

It follows that $\underline{G^2 = F}$.

4.a) "$\Rightarrow$": Assume that $F$ is injective, and let $x \in \mathbb{R}^m$ be a vector such that $F(x) = 0$.
Since $F(0) = 0$, we have $F(x) = F(0)$, which implies that $x = 0$ because $F$ is injective.
So $\underline{\text{the equation } F(x) = 0 \text{ has unique solution } x = 0}$.

"$\Leftarrow$": Assume that the equation $F(x) = 0$ has unique solution $x = 0$.
Let $u, v \in \mathbb{R}^m$ and assume that $F(u) = F(v)$.
Then, because $F$ is linear,
$0 = F(u) - F(v) = F(u - v) \implies u - v = 0 \implies u = v$.
Hence, $\underline{F \text{ is injective}}$.

b)

"$\Leftarrow$": If $F$ is bijective then it is <u>injective</u> by definition.

"$\Rightarrow$": Assume that $F$ is injective. Then, by (a), the equation $F(x)=0$ has unique solution $x=0$.

Let $A=[F]$ and $B=\mathrm{rref}(A)$.
The augmented matrix of the equation $F(x)=0$ is $(A|0)\sim(B|0)$, and so $F(x)=0 \Leftrightarrow Ax=0 \Leftrightarrow Bx=0$.
Since $B$ is on row-reduced echelon form, the equation $Bx=0$ having unique solution means that $B$ has a pivot element in every column.
As $B\in\mathbb{R}^{n\times n}$ is a square matrix, this implies that $B=I_n$.

Let $y\in\mathbb{R}^n$. The augmented matrix of the equation $F(x)=y$ is

$(A|y)\sim(B|z)=(I_n|z)$ for some $z\in\mathbb{R}^n$.

Hence, $F(x)=y \Leftrightarrow Ax=y \Leftrightarrow I_n x=z \Leftrightarrow x=z$.

In other words, the equation $F(x)=y$ has a unique solution $x=z$ for any $y\in\mathbb{R}^n$, that is, <u>$F$ is bijective</u>.