Linear Algebra I, autumn term 2018
Solutions to final exam 4th February 2019

1.a) For matrices $X \in \mathbb{R}^{l \times m}$, $Y \in \mathbb{R}^{n \times p}$, the product XY is defined if and only if $m = n$ (that is, the number of columns in X equals the number of rows in Y).
Therefore, the product AB is defined, but BA is not defined.

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 1 & -5 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & 6 & -1 & 7 \\ 1 & 4 & 2 & 5 \\ 3 & 1 & -6 & 2 \end{pmatrix}}}$$

b) A matrix $X \in \mathbb{R}^{l \times m}$ is invertible if and only if $l = m = \operatorname{rank}(X)$.
Hence, in particular, B is not invertible.

To determine if A is invertible, consider the following augmented matrix:

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 2 & 0 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -2 & 2 & 1 \\ 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 0 & -1 & -3 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & -2 & -1 \\ 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & 3 & -4 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & 3 & -4 & -1 \end{array}\right) = (I_3 \mid A^{-1})$$

It follows that A is invertible, and $A^{-1} = \underline{\underline{\begin{pmatrix} -2 & 3 & 1 \\ 2 & -2 & -1 \\ 3 & -4 & -1 \end{pmatrix}}}$.

2.a) $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A)$

pivot elements

<u>ker F</u>: The augmented matrix of the equation $F(x) = 0$ is

$(A|0) \sim (\mathrm{rref}(A)|0) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ by previous calculation.

Hence $x \in \ker F \iff \begin{cases} x_1 - x_4 = 0 \\ x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \end{cases}$ Setting $x_4 = t$ we get

$x = \begin{pmatrix} t \\ -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$

Hence $\ker F = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}\right\}$, and the vector $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$ constitutes a basis of $\ker F$.

<u>im F</u>: Since rref(A) has pivot elements in the first three columns,

the vectors $Ae_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $Ae_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ and $Ae_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ form a basis of im F.

b) Let $\underline{v} = (v_1, v_2, v_3)$, where $v_1 = Ae_1$, $v_2 = Ae_2$, $v_3 = Ae_3$ be the basis from (a). Perform the Gram-Schmidt algorithm on this basis to obtain an orthonormal basis of $\operatorname{im} F$.

· First, set $\underline{f_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}}$.

· Second, let $w_2 = v_2 - (v_2 \cdot f_1) f_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \frac{5}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$,

and $\underline{f_2 = \frac{1}{\|w_2\|} w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}$.

· Third, set $w_3 = v_3 - (v_3 \cdot f_1) f_1 - (v_3 \cdot f_2) f_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 5 \end{pmatrix}$, and $\underline{f_3 = \frac{1}{\|w_3\|} w_3 = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 5 \end{pmatrix}}$.

<u>Then (f_1, f_2, f_3) is an orthonormal basis of $\operatorname{im} F$.</u>

c) $P_{\operatorname{im} F}(v) = (v \cdot f_1) f_1 + (v \cdot f_2) f_2 + (v \cdot f_3) f_3 = \sqrt{5} f_1 + 2 f_2 + 0 f_3$

$= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} = v.$

Now, $v = P_{\operatorname{im} F}(v) \in \operatorname{im} F \implies \underline{v \in \operatorname{im} F}$, and
$v = \sqrt{5} f_1 + 2 f_2 + 0 f_3 \implies \underline{[v]_{\underline{f}} = \begin{pmatrix} \sqrt{5} \\ 2 \\ 0 \end{pmatrix}}$

3.a) $F(u)=u \iff F(u)-u=0 \iff Bu-I_3u=0 \iff (B-I_3)u=0$
$\iff u\in\ker(B-I_3)$

$$B-I_3=\begin{pmatrix}1&1&2\\3&3&6\\-2&-2&-4\end{pmatrix}\sim\begin{pmatrix}1&1&2\\0&0&0\\0&0&0\end{pmatrix}=\mathrm{rref}(B-I_3)$$

Hence, $u\in\ker(B-I_3)\iff u_1+u_2+2u_3=0$. Set $u_2=s$, $u_3=t$:

$$u=\begin{pmatrix}-s-2t\\s\\t\end{pmatrix}=s\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-2\\0\\1\end{pmatrix},\quad s,t\in\mathbb{R},$$ and hence

$\ker(B-I_3)=\mathrm{span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$. The vectors $\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-2\\0\\1\end{pmatrix}$ are linearly independent, so $\dim(\ker(B-I_3))=2$.

This shows that $\underline{U=\ker(B-I_3)=\mathrm{span}}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$ is a subspace satisfying that $\dim U=2$ and $F(u)=u$ for all $u\in U$.

b) If $\underline{v}=(v_1,v_2,v_3)$ is such a basis, then $v_1,v_2\in U=\ker(B-I_3)$, whilst

$F(v_3)=v_3+v_2\Rightarrow v_3\notin U$.

Now, for example, $(B-I_3)e_1=\begin{pmatrix}1\\3\\-2\end{pmatrix}=3\begin{pmatrix}-1\\1\\0\end{pmatrix}-2\begin{pmatrix}-2\\0\\1\end{pmatrix}\in U$, so $e_1\notin U$,

and $F(e_1)=e_1+\begin{pmatrix}1\\3\\-2\end{pmatrix}$, where $\begin{pmatrix}1\\3\\-2\end{pmatrix}\in U$.

Take $\underline{v_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\ v_2=\begin{pmatrix}1\\3\\-2\end{pmatrix},\ v_3=\begin{pmatrix}1\\0\\0\end{pmatrix}}$. Then $F(v_1)=v_1$, $F(v_2)=v_2$ and $F(v_3)=v_2+v_3$.

It remains to show that (v_1, v_2, v_3) is a basis of $\mathbb{R}^3$.

Assume that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.
This equations has augmented matrix

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$ so $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and v_1, v_2, v_3 are linearly independent.

Since $\dim \mathbb{R}^3 = 3$, it follows that $\underline{\underline{v = (v_1, v_2, v_3) \text{ is a basis of } \mathbb{R}^3.}}$

Moreover, $[F(v_1)]_{\underline{v}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[F(v_2)]_{\underline{v}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $[F(v_3)]_{\underline{v}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \underline{\underline{[F]_{\underline{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}}$

4) $(Ax)\cdot(Ay) = (Ax)^T(Ax) = x^T A^T A y = x \cdot (A^T A y)$

Hence $(Ax)\cdot(Ay) = x \cdot y \iff x \cdot (A^T A y) = x \cdot y$

- If $A^T A = I_m$ then clearly $\underline{\underline{x \cdot y = x \cdot (A^T A y) = (Ax)\cdot(Ay)}}$ for all $x, y \in \mathbb{R}^m$.

- Conversely, assume that $x \cdot y = (Ax)\cdot(Ay)$ for all $x, y \in \mathbb{R}^m$, and let

$$A^T A = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix} \in \mathbb{R}^{m \times m}.$$ Then, for all $1 \le i, j \le m$:

$$b_{ij} = e_i^T A^T A e_j = (Ae_i)\cdot(Ae_j) = e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \Longrightarrow \underline{\underline{A^T A = I_m}}.$$