Linear Algebra I, autumn term 2017
Solutions to homework 4

1) Clearly, $+ : \mathcal{F}(\mathbb{R},\mathbb{R}) \times \mathcal{F}(\mathbb{R},\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R},\mathbb{R})$, $(f,g) \longmapsto f+g$
and $\cdot : \mathbb{R} \times \mathcal{F}(\mathbb{R},\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R},\mathbb{R})$, $(\lambda, f) \longmapsto \lambda f$
where $\begin{cases} (f+g)(t) = f(t) + g(t) \\ (\lambda f)(t) = \lambda \cdot f(t) \end{cases} \quad \forall t \in \mathbb{R}$
are well-defined functions.

Axioms: Let $f, g, h \in \mathcal{F}(\mathbb{R},\mathbb{R})$, $\lambda, \mu \in \mathbb{R}$, and let $n \in \mathcal{F}(\mathbb{R},\mathbb{R})$ be the zero function
(i.e., $n(t) = 0$ for all $t \in \mathbb{R}$).

(i) $\big((f+g)+h\big)(t) = (f+g)(t) + h(t) = f(t) + g(t) + h(t) = f(t) + (g+h)(t) = \big(f+(g+h)\big)(t)$
for all $t \in \mathbb{R}$ $\Longrightarrow$ $\underline{(f+g)+h = f+(g+h)}$

(ii) $(f+g)(t) = f(t) + g(t) = g(t) + f(t) = (g+f)(t)$, $\forall t \in \mathbb{R}$ $\Longrightarrow$ $\underline{f+g = g+f}$

(iii) $\forall t \in \mathbb{R}$: $(n+f)(t) = n(t) + f(t) = f(t)$ $\Longrightarrow$ $\underline{n+f = f}$

(iv) The function $-f = (-1)f$ satisfies $\big(f+(-f)\big)(t) = f(t) + (-f)(t) = f(t) - f(t)$
$= 0 = n(t)$ $\forall t \in \mathbb{R}$ $\Longrightarrow$ $\underline{f+(-f) = n}$

(v) $\big(\lambda \cdot (f+g)\big)(t) = \lambda \cdot (f+g)(t) = \lambda\big(f(t) + g(t)\big) = \lambda f(t) + \lambda g(t) = (\lambda f)(t) + (\lambda g)(t) = (\lambda f + \lambda g)(t)$
$\Longrightarrow$ $\underline{\lambda(f+g) = \lambda f + \lambda g}$

(vi) $((\lambda+\mu)\cdot f)(t) = (\lambda+\mu)f(t) = \lambda f(t) + \mu f(t) = (\lambda f)(t) + (\mu f)(t) = (\lambda f + \mu f)(t)$

$\Longrightarrow \underline{(\lambda+\mu)f = \lambda f + \mu f}$

(vii) $(\lambda(\mu f))(t) = \lambda((\mu f)(t)) = \lambda(\mu \cdot f(t)) = (\lambda\mu)f(t) = ((\lambda\mu)f)(t)$

$\Longrightarrow \underline{\lambda(\mu f) = (\lambda\mu)f}$

(viii) $(1\cdot f)(t) = 1\cdot f(t) = f(t) \Longrightarrow \underline{1\cdot f = f}$

Since the axioms (i) – (viii) are satisfied, $\mathcal{F}(\mathbb{R},\mathbb{R})$ is a vector space with the given operations.

2.a) · $\underline{n \in \mathcal{P}}$ (the zero function is a polynomial)

· If $f(t) = \sum_{i=0}^{m} a_i t^i$ and $g(t) = \sum_{j=0}^{m} b_i t^i$ are polynomials,

then $(f+g)(t) = f(t) + g(t) = \sum_{i=0}^{m} a_i t^i + \sum_{j=0}^{m} b_i t^i = \sum_{i=0}^{m} (a_i + b_i) t^i$ is again a polynomial, i.e., $\underline{f+g \in \mathcal{P}}$.

· If $f(t) = \sum_{i=0}^{m} a_i t^i$ and $\lambda \in \mathbb{R}$ then $(\lambda f)(t) = \lambda \sum_{i=0}^{m} a_i t^i = \sum_{i=0}^{m} (\lambda a_i) t^i$

$\Longrightarrow \underline{\lambda f \in \mathcal{P}}.$

Hence, $\mathcal{P} \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ is a subspace.

b) $n \notin \mathcal{P}_m$, so $\mathcal{P}_m \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ is not a subspace.

c) Let f be the constant function $f(t) = -1$, $\forall t \in \mathbb{R}$.
Then $f(x) = -1$ so $f \in U_x$.
Now $((-1)\cdot f)(x) = -f(x) = -(-1) = 1 > 0 \Rightarrow (-1)\cdot f \notin U_x$.
Hence $U_x \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ is not a subspace.

d) $\cdot\, n \in V_x$, as $n(x) = 0$.

$\cdot$ If $f, g \in V_x$ then $f(x) = g(x) = 0$

$\Rightarrow (f+g)(x) = f(x) + g(x) = 0 + 0 = 0 \Longrightarrow f+g \in V_x$.

$\cdot$ If $f \in V_x$, $\lambda \in \mathbb{R}$, then $(\lambda f)(x) = \lambda f(x) = \lambda \cdot 0 = 0 \Rightarrow \lambda f \in V_x$.

Hence $V_x \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ is a subspace.

3) $v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $F: \mathbb{R}^{2\times 2} \to \mathbb{R}^2$, $A \mapsto Av$

a) Let $\begin{cases} A, B \in \mathbb{R}^{2\times 2} \\ \lambda \in \mathbb{R} \end{cases}$. Then $F(A+B) = (A+B)v = Av + Bv = F(A) + F(B)$

$$F(\lambda A) = (\lambda A)v = \lambda(Av) = \lambda F(A)$$

$\Longrightarrow$ <u>F is linear</u>.

b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $F(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -a+2b \\ -c+2d \end{pmatrix}$

$F(A) = 0 \iff \begin{cases} -a+2b = 0 \\ -c+2d = 0 \end{cases} \iff \begin{cases} a = 2b \\ c = 2d \end{cases}$.

Set $b = s$, $c = t$. Then $F(A) = 0 \iff \begin{cases} a = 2s \\ b = s \\ c = 2t \\ d = t \end{cases}$

$$\iff A = \begin{pmatrix} 2s & s \\ 2t & t \end{pmatrix} = s\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

$\Rightarrow \ker F = \operatorname{span}\left\{ \underset{A_1}{\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{A_2}{\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}} \right\}$.

If $\lambda_1 A_1 + \lambda_2 A_2 = 0$: $0 = \lambda_1 \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 & \lambda_1 \\ 2\lambda_2 & \lambda_2 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0$

Hence A_1, A_2 are linearly independent.

$\Rightarrow$ <u>$\mathcal{A} = (A_1, A_2)$ is a basis of $\ker F$</u>.

c) $F(B) = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2+2 \\ -4+4 \end{pmatrix} = 0 \Rightarrow \underline{B \in \ker F}$.

$\lambda_1 A_1 + \lambda_2 A_2 = B \iff \begin{pmatrix} 2\lambda_1 & \lambda_1 \\ 2\lambda_2 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \iff \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$

$\Rightarrow \underline{\underline{[B]_{\underline{A}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}}$.

4) Let $u \in V$. Then $\underline{0u} = (0+0)u \overset{(vi)}{=} \underline{0u + 0u}$. ⊗

By axiom (iv), there exists a $w \in V$ s.th. $0u + w = \mathbf{0}$.

Now $\mathbf{0} = 0u + w \overset{⊗}{=} (0u+0u) + w \overset{(i)}{=} 0u + (0u+w) = 0u + \mathbf{0} \overset{(iii)}{=} 0u$

$\underline{\underline{\text{Hence, } 0u = \mathbf{0}.}}$ (**)

Next, $u + (-1)u \overset{(viii)}{=} 1u + (-1)u \overset{(vi)}{=} (1+(-1))u = 0u \overset{(**)}{=} \mathbf{0}$

$\Rightarrow \underline{\underline{u + (-1)u = \mathbf{0}}}$.