Linear algebra I, autumn term 2017
Solutions to Homework 3

1. a)

$$Ax = \begin{pmatrix} 1 & -2 & 1 & -3 \\ 1 & -2 & 2 & -1 \\ 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4+6-4-6 \\ 4+6-8-2 \\ 4+6-12+2 \\ 0+0-8+8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \underline{\underline{x \in \ker A}}$$

$$Ay = \begin{pmatrix} 1 & -2 & 1 & -3 \\ 1 & -2 & 2 & -1 \\ 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1-2+3-12 \\ -1-2+6-4 \\ -1-2+9+4 \\ 0+0+6+16 \end{pmatrix} = \begin{pmatrix} -12 \\ -1 \\ 10 \\ 22 \end{pmatrix} \neq 0 \Longrightarrow \underline{\underline{y \notin \ker A}}$$

$$(A \mid x \; y) = \left(\begin{array}{cccc|cc} 1 & -2 & 1 & -3 & 4 & -1 \\ 1 & -2 & 2 & -1 & -3 & 1 \\ 1 & -2 & 3 & 1 & -4 & 3 \\ 0 & 0 & 2 & 4 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 1 & -2 & 1 & -3 & 4 & -1 \\ 0 & 0 & 1 & 2 & -7 & 2 \\ 0 & 0 & 2 & 4 & -8 & 4 \\ 0 & 0 & 2 & 4 & 2 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cc} 1 & -2 & 0 & -5 & 11 & -3 \\ 0 & 0 & 1 & 2 & -7 & 2 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 \end{array}\right)$$

$\Longrightarrow$

$(A|x) \sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 11 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 16 \end{array}\right) \Longrightarrow$ The eq. $Au = x$ has no solution $u \in \mathbb{R}^4$ $\Longrightarrow \underline{\underline{x \notin \operatorname{im} A}}$

$(A|y) \sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -5 & -3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow$ The eq. $Au = y$ has a solution $u \in \mathbb{R}^4$ $\Longrightarrow \underline{\underline{y \in \operatorname{im} A}}$

1.b) By previous calculation, $A \sim \begin{pmatrix} 1 & -2 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A)$

Hence, $Au = 0 \iff \begin{cases} u_1 - 2u_2 - 5u_4 = 0 \\ u_3 + 2u_4 = 0 \end{cases}$

Set $u_2 = s$, $u_4 = t$. $Au = 0 \iff u = \begin{pmatrix} 2s+5t \\ s \\ -2t \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$, $s,t \in \mathbb{R}$

$\Rightarrow \ker A = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$, and $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$ are linearly independent.

$\Rightarrow \overset{v_1}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 5 \\ 0 \\ -2 \\ 1 \end{pmatrix}}$ form a basis of $\ker A$.

As $\operatorname{rref}(A)$ has pivot elements in the first and third columns, it follows that Ae_1 and Ae_3 form a basis of $\operatorname{im} A$.

I.e., $\overset{w_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}}$ form a basis of $\operatorname{im} A$.

c) The eq. $\lambda_1 v_1 + \lambda_2 v_2 = x$ has augm. matrix $\left(\begin{array}{cc|c} 2 & 5 & 4 \\ 1 & 0 & -3 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 2 & 5 & 4 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{array}\right)$

$\sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 5 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\Rightarrow \lambda_1 = -3$, $\lambda_2 = 2$.

$\Rightarrow [x]_{\underline{v}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$[y]_{\underline{v}}$ is not defined, since $y \notin \ker A$

By the calculation in (1.a), we have

$$(A|y) \sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -5 & -3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} | & | & | \\ w_1 & w_2 & y \\ | & | & | \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ The eq. $\mu_1 w_1 + \mu_2 w_2 = y$ has solution $\begin{cases} \mu_1 = -3 \\ \mu_2 = 2 \end{cases}$

$\Rightarrow$ $\underline{\underline{[y]_{\underline{w}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}}}$. $\underline{[x]_{\underline{w}} \text{ is not defined, since } x \notin \operatorname{im} A}$.

2) The equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0$ has augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 3 & 0 & 2 & 0 \\ 3 & 4 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 2 & -4 & 0 \\ 0 & -2 & 4 & -8 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & -5 & 0 \\ 0 & -1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & -5 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

As the row-reduced echelon form has pivot elements in the first and second columns, it follows that $\underline{\underline{(u_1, u_2) = \underline{u}}}$ is a basis of U.

Moreover, $\left(\begin{array}{cc|c} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \underline{\underline{[u_3]_{\underline{u}} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}}}$, and

$$\left(\begin{array}{cc|c} | & | & | \\ u_1 & u_2 & u_4 \\ | & | & | \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \underline{\underline{[u_4]_{\underline{u}} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}}}$$

Clearly, $\underline{\underline{[u_1]_{\underline{u}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}}$ and $\underline{\underline{[u_2]_{\underline{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$.

3) $Av_1 = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = v_1 \Rightarrow [Av_1]_{\underline{v}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$Av_2 = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = v_2 \Rightarrow [Av_2]_{\underline{v}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$Av_3 = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 \\ 8 \\ 11 \end{pmatrix}$

The equation $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{pmatrix} 5 \\ 8 \\ 11 \end{pmatrix}$ has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 0 & 1 & 8 \\ 1 & 1 & 2 & 11 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow \begin{cases} \lambda_1 = 8 \\ \lambda_2 = 3 \\ \lambda_3 = 0 \end{cases} \Rightarrow \left[\begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}\right]_{\underline{v}} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix} \Rightarrow [Av_3]_{\underline{v}} = \left[\frac{1}{6}\begin{pmatrix} 5 \\ 8 \\ 11 \end{pmatrix}\right]_{\underline{v}} = \frac{1}{6}\begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$

Hence, $[F]_{\underline{v}} = \begin{pmatrix} | & | & | \\ [Av_1]_{\underline{v}} & [Av_2]_{\underline{v}} & [Av_3]_{\underline{v}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 8/6 \\ 0 & 1 & 3/6 \\ 0 & 0 & 0 \end{pmatrix} = \underline{\underline{\frac{1}{6}\begin{pmatrix} 6 & 0 & 8 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{pmatrix}}}$

4) Let $x \in V+W$. Then $x = y+z$ for some $y \in V$, $z \in W$.

Since $\underline{v}$ and $\underline{w}$ are bases of V respectively W, there exist scalars $\lambda_i, (i=1,\dots,\ell)$ and μ_j $(j=1,\dots,m)$ such that $y = \sum_{i=1}^{\ell} \lambda_i v_i$ and $z = \sum_{j=1}^{m} \mu_j w_j$.

$$\Longrightarrow \quad x = y+z = \sum_{i=1}^{\ell} \lambda_i v_i + \sum_{j=1}^{m} \mu_j w_j \in \operatorname{span}\{v_1,\dots,v_\ell, w_1,\dots,w_m\}$$

$$\Longrightarrow \quad V+W \subset \operatorname{span}\{v_1,\dots,v_\ell, w_1,\dots,w_m\}.$$

On the other hand, $v_1,\dots,v_\ell, w_1,\dots,w_m \in V+W \Longrightarrow \operatorname{span}\{v_1,\dots,v_\ell,w_1,\dots,w_m\} \subset V+W$ by the Remark in Section 8 of the lectures.

$$\Longrightarrow \quad \underline{V+W = \operatorname{span}\{v_1,\dots,v_\ell, w_1,\dots,w_m\}}$$

$(*)$

Assume that $\underline{\alpha_1 v_1 + \dots + \alpha_\ell v_\ell + \beta_1 w_1 + \dots + \beta_m w_m = 0}$ for some $\alpha_1,\dots,\alpha_\ell, \beta_1,\dots,\beta_m \in \mathbb{R}$.

Then $\alpha_1 v_1 + \dots + \alpha_\ell v_\ell = (-\beta_1) w_1 + \dots + (-\beta_m) w_m \in V \cap W = \{0\}$

$$\Longrightarrow \begin{cases} \alpha_1 v_1 + \dots + \alpha_\ell v_\ell = 0 \Longrightarrow \alpha_1 = \dots = \alpha_\ell = 0 \text{ since } v_1,\dots,v_\ell \text{ are lin. indep.} \\ \beta_1 w_1 + \dots + \beta_m w_m = 0 \Longrightarrow \beta_1 = \dots = \beta_m = 0 \text{ since } w_1,\dots,w_m \text{ are lin. indep.} \end{cases}$$

As the equation $(*)$ has unique solution $\alpha_1 = \dots = \alpha_\ell = \beta_1 = \dots = \beta_m = 0$, the vectors $\underline{v_1,\dots,v_\ell, w_1,\dots,w_m \text{ are linearly independent}}$.

$\underline{\text{Hence } (v_1,\dots,v_\ell, w_1,\dots,w_m) \text{ is a basis of } V+W}$