Linear algebra I, autumn term 2017
Solutions to Homework 2

1) Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1/\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix}$. Then, for all $x \in \mathbb{R}^2$,

$$Ax = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1/\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 \\ 1/\sqrt{2}\, x_1 + 2x_2 \\ 0 \end{pmatrix} = F(x).$$

This implies that $\underline{F \text{ is linear}}$, and $\underline{[F] = A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1/\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix}}$

The map G is $\underline{\text{not linear}}$: $G(2e_2) = G\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2e_2 = 2G(e_2)$

For all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$:

$$H_u(x+y) = u\cdot(x+y) = u\cdot x + u\cdot y = H_u(x) + H_u(y), \text{ and}$$

$$H_u(\lambda x) = u\cdot(\lambda x) = \lambda(u\cdot x) = \lambda H_u(x)$$

$\longrightarrow$ $\underline{H_u \text{ is linear}}$

For all $i \in \{1, 2, \dots, n\}$: $H_u(e_i) = u\cdot e_i = u_i \in \mathbb{R}$

$$\Longrightarrow \underline{[H_u] = (u_1 \; u_2 \cdots u_n) \in \mathbb{R}^{1\times n}}$$

2) $P_u(e_1) = \frac{u \cdot e_1}{u \cdot u} u = -\frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$P_u(e_2) = \frac{u \cdot e_2}{u \cdot u} u = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow [P_u] = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$[rot_\alpha] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$[rot_\alpha \circ rot_\beta] = [rot_\alpha][rot_\beta] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \left(= [rot_{\alpha+\beta}]\right)$$

$$\Rightarrow [rot_\beta \circ rot_\alpha] = \begin{pmatrix} \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{pmatrix} = [rot_\alpha \circ rot_\beta]$$

$$[rot_\alpha \circ P_u] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\alpha + \sin\alpha & -(\cos\alpha + \sin\alpha) \\ \sin\alpha - \cos\alpha & -\sin\alpha + \cos\alpha \end{pmatrix}$$

$$[P_u \circ rot_\alpha] = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos\alpha - \sin\alpha & -\sin\alpha - \cos\alpha \\ -\cos\alpha + \sin\alpha & \sin\alpha + \cos\alpha \end{pmatrix}$$

[Note that $P_u \circ rot_\alpha$ and $rot_\alpha \circ P_u$ are not equal, in general].

3) Let $B=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Then $AB-BA=\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}-\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$=\begin{pmatrix} a-2b & c-2d \\ 2a+b & 2c+d \end{pmatrix}-\begin{pmatrix} a+2c & -2a+c \\ b+2d & -2b+d \end{pmatrix}=\begin{pmatrix} -2b-2c & 2a-2d \\ 2a-2d & 2c+2b \end{pmatrix}$

Now $AB=BA \iff AB-BA=0 \iff \begin{cases} -2b-2c=0 \\ 2a-2d=0 \\ 2a-2d=0 \\ 2c+2b=0 \end{cases}$

$\iff \begin{cases} 2a-2d=0 \\ 2c+2b=0 \end{cases} \iff \begin{cases} a=d \\ b=-c \end{cases} \iff \underline{\underline{B=\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \text{ for some } \lambda,\mu\in\mathbb{R}}}$.

4) Let $A=[F]$. $\operatorname{rk} F=2 \iff \operatorname{rref}(A)$ has two pivot elements.

$\Rightarrow$ The linear system $Ax=0$ has two pivot variables

$\Rightarrow$ One of the variables in the system $Ax=0$ is not a pivot variable

$\Rightarrow$ The system $Ax=0$ has infinitely many solutions, that is,

there exist infinitely many $x\in\mathbb{R}^3$ s.th. $F(x)=Ax=0$, QED

5.a) For all $x \in \mathbb{R}^n$, $[P_u]x = \frac{1}{u \cdot u} u \underbrace{u^T x}_{\in \mathbb{R}} = \frac{u^T x}{u \cdot u} u = \frac{u \cdot x}{u \cdot u} u = P_u(x)$.

b) $P_u(x) = 0 \iff \frac{u \cdot x}{u \cdot u} u = 0 \overset{(u \neq 0)}{\iff} \frac{u \cdot x}{u \cdot u} = 0 \iff u \cdot x = 0$.

c) If $n=1$ then $u \in \mathbb{R}, u \neq 0$ is a non-zero number, and $P_u(x) = \frac{ux}{u^2} u = x$ so $P_u(x) = x$ for all x. Hence <u>P_u is invertible in this case.</u>

If $n > 1$ then there exist $x \in \mathbb{R}^n$, $x \neq 0$ s.th. $u \cdot x = 0$ and thus $P_u(x) = 0 = P_u(0)$. Therefore, <u>P_u is not invertible for $n > 1$</u>.