Linear algebra I, autumn term 2017

Solutions to midterm exam the 27^{th} November 2017

1a) The augmented matrix of the system $Ax = b$ is

$$(A|b) = \left(\begin{array}{cccc|c} 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 3 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 3 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -1 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & 7/2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right) \qquad Ax = b \iff \underline{\underline{x = \begin{pmatrix} -1 \\ 3 \\ -1/2 \\ 1/2 \end{pmatrix}}}$$

b) From the above calculation, it follows that

$$(A|0) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$ Hence the eq. $Ax = 0$ has

the unique solution $\underline{\underline{x = 0}}$.

c) Since $A \sim I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the row-reduced echelon form of A is $\underline{\underline{I_4}}$.

$$B \sim \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = rref(B)}}.$$

d) $A \in \mathbb{R}^{4\times 4}$, $\mathrm{rank}(A) = 4 \implies$ $\underline{\underline{A \text{ is invertible}}}$.

The matrix B is <u>not invertible</u>, since it is not a square matr.

e) To satisfy the equation, X needs to have size 4×2.

Denote the columns of X by v respectively w: $X = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}$.

Then $AX = B \iff [Av = 0 \text{ and } Aw = b]$

$$\overset{\text{by (a,b)}}{\iff} v = 0, \quad w = \begin{pmatrix} -1 \\ 3 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$\iff \underline{\underline{X = \begin{pmatrix} 0 & -1 \\ 0 & 3 \\ 0 & -1/2 \\ 0 & 1/2 \end{pmatrix}}}.$$

2.a) Let $A=\begin{pmatrix}1&0&-3\\0&1&0\\1&0&0\end{pmatrix}$. Then

$$Ax=\begin{pmatrix}1&0&-3\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1-3x_3\\x_2\\x_1\end{pmatrix}=F(x) \text{ for all } x\in\mathbb{R}^3,$$

implying that $\underline{F:\mathbb{R}^3\to\mathbb{R}^3 \text{ is a linear map}}$, and $\underline{[F]=A}$.

Let $\lambda=2$. Then
$$q(\lambda e_1)=q\begin{pmatrix}2\\0\\0\end{pmatrix}=2^2=4$$
$$2q(e_1)=2q\begin{pmatrix}1\\0\\0\end{pmatrix}=2\cdot 1^2=2$$
$\Longrightarrow$ $\underline{q \text{ is not linear.}}$

b) By (a), the matrix of F is $\underline{[F]=A=\begin{pmatrix}1&0&-3\\0&1&0\\1&0&0\end{pmatrix}}$.

3.a) $e_1 = \lambda_1 v_1 + \lambda_2 v_3 \iff \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This is a linear system with augmented matrix $\left(\begin{array}{cc|c} | & | & | \\ v_1 & v_2 & e_1 \\ | & | & | \end{array}\right)$.

Similarly the eq. $e_2 = \mu_1 v_1 + \mu_2 v_2$ gives a system with augm. matrix $\left(\begin{array}{cc|c} | & | & | \\ v_1 & v_2 & e_2 \\ | & | & | \end{array}\right)$. We solve these two systems together:

$$\left(\begin{array}{cc|cc} | & | & | & | \\ v_1 & v_2 & e_1 & e_2 \\ | & | & | & | \end{array}\right) = \begin{array}{c} \\ -2 \end{array}\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \sim \begin{array}{c} 2 \\ \end{array}\left(\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \begin{array}{c} -1 \\ \end{array}\left(\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 0 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\implies \left(\begin{array}{cc|c} | & | & | \\ v_1 & v_2 & e_1 \\ | & | & | \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right), \quad \left(\begin{array}{cc|c} | & | & | \\ v_1 & v_2 & e_2 \\ | & | & | \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right)$$

$$\implies \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -1 \end{cases}, \quad \begin{cases} \mu_1 = -3 \\ \mu_2 = 2 \end{cases}. \qquad \left[\begin{array}{l} e_1 = 2v_1 - v_2 \\ e_2 = -3v_1 + 2v_2 \end{array}\right]$$

b) $F(e_1) = F(2v_1 - v_2) = 2F(v_1) - F(v_2) = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$,

$F(e_2) = F(-3v_1 + 2v_2) = -3F(v_1) + 2F(v_2) = -3\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \end{pmatrix}$

$$\implies [F] = \begin{pmatrix} | & | \\ F(e_1) & F(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 1 & -1 \end{pmatrix}.$$

4a) $F_{\underline{v}}(x) = \sum_{i=1}^{m} x_i v_i$. For all $x, y \in \mathbb{R}^m$:

$$F_{\underline{v}}(x+y) = \sum_{i=1}^{m} (x_i + y_i) v_i = \sum_{i=1}^{m} (x_i v_i + y_i v_i) = \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{m} y_j v_j = F_{\underline{v}}(x) + F_{\underline{v}}(y)$$

For all $x \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$: $F_{\underline{v}}(\lambda x) = \sum_{i=1}^{m} (\lambda x_i) v_i = \sum_{i=1}^{m} \lambda (x_i v_i) = \lambda \sum_{i=1}^{m} x_i v_i = \lambda F_{\underline{v}}(x)$

$\Longrightarrow$ $\underline{F_{\underline{v}} \text{ is a linear map.}}$

b) $w = \sum_{i=1}^{m} \lambda_i v_i \iff w = F_{\underline{v}} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}$.

Hence, $\underline{v}$ is a basis if and only if for all $w \in \mathbb{R}^n$, the equation $w = F_{\underline{v}} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$ has a unique solution $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \in \mathbb{R}^m$, that is, if and only if F is invertible.

c) If $\underline{v} = (v_1, \ldots, v_m)$ is a basis then $F_{\underline{v}} : \mathbb{R}^m \to \mathbb{R}^n$ is invertible, hence, in particular, $\underline{\underline{m = n}}$ (and $\operatorname{rank}(F_{\underline{v}}) = m$).