Linear Algebra I, autumn term 2017
Solutions to final exam, 5th February 2018

1.a)

$$F(x) = Ax = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 4 & 5 & 3 \\ 2 & 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-2+1-1 \\ 2-3+3-2 \\ 2-4+5-3 \\ 4-5+4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \underline{x \in \ker F}$$

b) Find the row-reduced echelon form of the matrix A:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 4 & 5 & 3 \\ 2 & 5 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \underbrace{\begin{pmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{= \operatorname{rref}(A)}$$

The equation $F(x) = 0$ has augmented matrix $(A|0) \sim (\operatorname{rref}(A)|0)$, hence its solutions are:

$$\begin{cases} x_1 = 3s + t \\ x_2 = -2s - t \\ x_3 = s \\ x_4 = t \end{cases}, \quad s, t \in \mathbb{R}, \quad \text{that is,} \quad x = s\begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

$\rightarrow$ The vectors $\begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\ker F$, and $\dim(\ker F) = 2$.

Since $\operatorname{rref}(A)$ has pivot elements in the first and second colums, the vectors $F(e_1)$ and $F(e_2)$ form a basis of $\operatorname{im} F$:

$\underline{v} = (v_1, v_2)$, where $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, is a basis of $\operatorname{im} F$, $\dim(\operatorname{im} F) = 2$.

c) $F(e_1) = v_1 \implies [F(e_1)]_{\underline{v}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$F(e_2) = v_2 \implies [F(e_2)]_{\underline{v}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The equation $\lambda_1 v_1 + \lambda_2 v_2 = F(e_3)$ has augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right). \text{ Hence } [F(e_3)]_{\underline{v}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

by the computation of $rref(A)$

The equation $\lambda_1 v_1 + \lambda_2 v_2 = F(e_4)$ similarly has augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \\ 2 & 5 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies [F(e_4)]_{\underline{v}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2a) For all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$,

$$F(x) = \begin{pmatrix} -x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since F is given by multiplication by a matrix it is a linear map.

b) Set $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$F(v_1) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -v_1 \implies [F(v_1)]_{\underline{v}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$F(v_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_2 \implies [F(v_2)]_{\underline{v}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, $[F]_{\underline{v}} = \left([F(v_1)]_{\underline{v}} \; [F(v_2)]_{\underline{v}} \right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3a) $U = \operatorname{span}\{e_1, e_2\} \subset \mathbb{R}^3$.

The vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent:

If $\lambda_1 e_1 + \lambda_2 e_2 = 0$, then

$$0 = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \implies \lambda_1 = \lambda_2 = 0 .$$

Hence $\dim U = 2$.

b) Let $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_1 + e_2 = \begin{pmatrix}1\\1\end{pmatrix} \in U$.

From (a), we know that $\underline{\operatorname{span}\{u_1, u_2\} = U}$.

Moreover, since $u_2 = u_3 - u_1$, we have $u_1, u_2 \in \operatorname{span}\{u_1, u_3\}$

$\Rightarrow U = \operatorname{span}\{u_1, u_2\} \subset \operatorname{span}\{u_1, u_3\} \subset U \Rightarrow \underline{\operatorname{span}\{u_1, u_3\} = U}$.

Similarly, $u_1 = u_3 - u_2$, so $U = \operatorname{span}\{u_1, u_2\} \subset \operatorname{span}\{u_2, u_3\} \subset U$
$\Rightarrow \underline{\operatorname{span}\{u_2, u_3\} = U}$.

c) Let $f(x) = \begin{cases} \begin{pmatrix}0\\0\end{pmatrix} & \text{if } x = 0, \\ \begin{pmatrix}1\\0\end{pmatrix} & \text{if } x \neq 0. \end{cases}$

Then $\operatorname{im} f = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \right\}$. Hence, $e_1 = \begin{pmatrix}1\\0\end{pmatrix} \in \operatorname{im} f$, but $2e_1 = \begin{pmatrix}2\\0\end{pmatrix} \notin \operatorname{im} f$,
<u>so $\operatorname{im} f$ is not a subspace of $\mathbb{R}^2$.</u>

4a) Let $v \in \mathbb{R}^n$. For all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$:

$$\left.\begin{aligned} v^*(x+y) &= v\cdot(x+y) = v\cdot x + v\cdot y = v^*(x) + v^*(y), \\ v^*(\lambda x) &= v\cdot(\lambda x) = \lambda(v\cdot x) = \lambda v^*(x) \end{aligned}\right\} \Longrightarrow v^* \text{ is linear, hence, } \underline{\underline{v^* \in (\mathbb{R}^n)^*}}.$$

b) Let $v, w \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then, $\varphi(v+w) = (v+w)^*$; and $\varphi(v) + \varphi(w) = v^* + w^*$.

For all $x \in \mathbb{R}^n$: $(v+w)^*(x) = (v+w)\cdot x = v\cdot x + w\cdot x = v^*(x) + w^*(x)$

$$\Longrightarrow (v+w)^* = v^* + w^*, \text{ i.e., } \varphi(v+w) = \varphi(v) + \varphi(w).$$

Moreover, $\varphi(\lambda v) = (\lambda v)^*$, and $(\lambda v)^*(x) = (\lambda v)\cdot x = \lambda(v\cdot x) = \lambda v^*(x)$

$$\Longrightarrow \varphi(\lambda v) = (\lambda v)^* = \lambda v^* = \lambda \varphi(v)$$

$\underline{\underline{\text{Hence, } \varphi \text{ is linear.}}}$

c) Set $\psi: (\mathbb{R}^n)^* \to \mathbb{R}^n$, $\psi(f) = \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}$ for all $f \in (\mathbb{R}^n)^*$.

Then, for any $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$:

$$\psi\varphi(v) = \psi(v^*) = \begin{pmatrix} v^*(e_1) \\ \vdots \\ v^*(e_n) \end{pmatrix} = \begin{pmatrix} v\cdot e_1 \\ \vdots \\ v\cdot e_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v$$

$$\Longrightarrow \underline{\psi\varphi = id_{\mathbb{R}^n}}.$$

For any $f \in (\mathbb{R}^n)^*$: $\varphi\psi(f) = \varphi\begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix} = \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}^*$.

For all $x = x_1 e_1 + \dots + x_n e_n \in \mathbb{R}^n$:

$$\begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}^* (x) = \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 f(e_1) + \dots + x_n f(e_n)$$

$$= f(x_1 e_1 + \dots + x_n e_n) = f(x)$$

$$\Longrightarrow \varphi\psi(f) = \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}^* = \mathrm{id}_{(\mathbb{R}^n)^*}$$

Hence, φ is invertible, with $\varphi^{-1} = \psi$.