Linear Algebra I, autumn term 2016
Solutions to the midterm exam, 21st November

1a) The augmented matrix of the system is:

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{pmatrix}}_{=A}\left|\begin{matrix} 1 \\ 2 \\ 2 \\ 1 \end{matrix}\right. \quad (-2)(-1) \sim \left(\begin{matrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}\left|\begin{matrix} 1 \\ 1 \\ 0 \\ 1 \end{matrix}\right.\right) \quad (1) \sim \left(\begin{matrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\left|\begin{matrix} 1 \\ 1 \\ 0 \\ 1 \end{matrix}\right.\right)$$

The fourth row corresponds to the equation $0=1$, hence the system has <u>no solution</u>.

b) By the previous calculation, we know that $A \sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence, $A \sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{(1)}{\sim} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{(2)}{\sim} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{(-1)(-1)}{\sim}$

$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A)$. Since $\operatorname{rref}(A)$ has 3 pivot element, <u>$\operatorname{rank}(A) = 3$</u>.

c) By the calculation in (b), we know that $\left(A \left| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right.\right) \sim \left(\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}\left|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right.\right)$.

Hence, $Ax = 0 \iff \begin{cases} x_1 + x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$ Set $x_4 = t$.

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

$\Longrightarrow$ <u>$\ker A = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right\}$</u>

2) Let $x = 1$. $F(0x) = F(0) = \begin{pmatrix}\cos 0\\ \sin 0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $0 \cdot F(x) = \begin{pmatrix}0\\0\end{pmatrix}$ $\Big\}$ $F(0x) \neq 0 \cdot F(x)$ so F is <u>not linear</u>.

Let $u = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$, $v = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$

$$G(u+v) = G\begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix} = \begin{pmatrix}u_2+v_2\\(u_1+v_1)+(u_2+v_2)\\0\\2(u_2+v_2)-3(u_3+v_3)\end{pmatrix} = \begin{pmatrix}u_2+v_2\\u_1+u_2+v_1+v_2\\0\\2u_2-3u_3+2v_2-3v_3\end{pmatrix}$$

$$= \begin{pmatrix}u_2\\u_1+u_2\\0\\2u_2-3u_3\end{pmatrix} + \begin{pmatrix}v_2\\v_1+v_2\\0\\2v_2-3v_3\end{pmatrix} = G(u) + G(v)$$

$$G(\lambda u) = G\begin{pmatrix}\lambda u_1\\\lambda u_2\\\lambda u_3\end{pmatrix} = \begin{pmatrix}\lambda u_2\\\lambda u_1+\lambda u_2\\0\\2\cdot\lambda u_2 - 3\cdot\lambda u_3\end{pmatrix} = \lambda\begin{pmatrix}u_2\\u_1+u_2\\0\\2u_2-3u_3\end{pmatrix} = \lambda G(u)$$

Hence, <u>G is a linear map</u>.

$$G(e_1) = G\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad G(e_2) = G\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\2\end{pmatrix},\quad G(e_3) = G\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\-3\end{pmatrix}$$

The matrix of G is $[G] = \begin{pmatrix}0&1&0\\1&1&0\\0&0&0\\0&2&-3\end{pmatrix}$.

3a) A^2 and B^2 are not defined, since A and B are not square matrices.
AB and BA are defined.

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix}}}$$

$$BA = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$$

b) A matrix $X \in \mathbb{R}^{m \times n}$ is invertible if and only if $m = n = \operatorname{rank} X$.

Since A and B are not square matrices, they do not satisfy this condition, so they are <u>not invertible</u>.

AB is <u>not invertible</u>, because its rank is less than 3:

$$AB = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(AB)$$

$\operatorname{rank}(AB) = 2 < 3$, so it is not invertible. two pivot elements

$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$. $I_2^2 = I_2$ so the matrix BA <u>is invertible</u>,

and $\underline{\underline{(BA)^{-1} = BA = I_2}}$

4) $F(v) = v \iff F(v) - v = 0$

$$F(v) - v = Av - v = Av - I_2 v = (A - I_2)v = \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/5 & 2/5 \\ 2/5 & -4/5 \end{pmatrix}$$

So the equation $F(v) - v = 0$ has augmented matrix

$$\left(\begin{array}{cc|c} -1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1/5 & 2/5 & 0 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad v_1 - 2v_2 = 0$$

Set $v_2 = t$.

$v = \begin{pmatrix} 2t \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for any $t \in \mathbb{R}$. Take, for example, $\underline{v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

$F(w) = 0 \iff Aw = 0$ this eq. has augmented matrix

$$\left(A \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right) = \left(\begin{array}{cc|c} 4/5 & 2/5 & 0 \\ 2/5 & 1/5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 4/5 & 2/5 & 0 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$w_1 + \frac{1}{2}w_2 = 0$. Set $w_2 = t$. $w = \begin{pmatrix} -\frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

Take, for example, $\underline{w = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$ $(t = -2)$

<u>The vector $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ satisfies $F(v) = v$, and $w = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ satisfies $F(w) = 0$</u>

b) Note that $v \cdot w = 0$, and that $v \cdot x = 0 \iff x = \lambda w$ for some $\lambda \in \mathbb{R}$.

Hence $F(x) = 0$ for all x that are orthogonal to v, and
$F(y) = y$ for all vectors y that are parallel to v.

<u>This means that $F = P_v$ - the orthogonal projection onto v.</u>

5a) For example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$Ae_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow e_1 \in \ker A$

$Ae_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \implies e_1 \in \operatorname{im} A.$

b) $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

c) Assume that $F^2 = 0$, that is, $F(F(x)) = 0$ for all $x \in \mathbb{R}^n$, and let $v \in \operatorname{im} F$. Then $v = F(u)$ for some $u \in \mathbb{R}^n$, so $F(v) = F(F(u)) = 0$ by the assumption. Hence $v \in \ker F$, and so $\operatorname{im} F \subset \ker F$.

Assume that $\operatorname{im} F \subset \ker F$, and let $x \in \mathbb{R}^n$.
Then $F^2(x) = F(\underbrace{F(x)}_{\in \operatorname{im} F \subset \ker F}) \Longrightarrow F(F(x)) = 0$

So $F^2(x) = F(F(x)) = 0$, and since x was an arbitrary vector in $\mathbb{R}^n$, this means that $F^2 = 0$.

d) Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ be such that $x \in \ker F$ and $x \in \operatorname{im} F$.

Set $[F] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Now $0 = F(x) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + cx_2 \\ bx_1 + dx_2 \end{pmatrix} = x_1 \begin{pmatrix} a \\ b \end{pmatrix} + x_2 \begin{pmatrix} c \\ d \end{pmatrix}$

If $x_1 = 0$: Then $x_2 \neq 0$, $x_2 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $[F] = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$

$\Rightarrow \operatorname{im} F = \operatorname{span}\left\{\begin{pmatrix} a \\ b \end{pmatrix}\right\}$.

If $x_1 \neq 0$: Then $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{x_2}{x_1}\begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow \operatorname{im} F = \operatorname{span}\left\{\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix} c \\ d \end{pmatrix}\right\}$.

In both cases, $\operatorname{im} F$ is spanned by a single vector.

Since $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{im} F$, $x \neq 0$, we get $\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$ if $x_1 = 0$, and

$\begin{pmatrix} c \\ d \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$ if $x_1 \neq 0$.

In both cases, $\operatorname{span}\{x\} = \operatorname{im} F$.

$$\left.\begin{array}{l} \text{Since } x \in \ker F, \text{ we have } F(tx) = tF(x) = 0 \text{ for all } t \in \mathbb{R} \\ \operatorname{im} F = \operatorname{span}\{x\} = \{tx \mid t \in \mathbb{R}\} \end{array}\right\} \Rightarrow F(y) = 0 \text{ for all } y \in \operatorname{im} F$$

$\Longrightarrow \operatorname{im} F \subset \ker F$.

Hence, $F^2 = 0$ by (c).