Linear Algebra I, autumn term 2016
Solutions to Homework 4

1.a) $\lambda_1 u_1 + \lambda_2 u_2 = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \iff \lambda_1 = \lambda_2 = 0$

This means that u_1, u_2 are linearly independent, and hence a basis of $\operatorname{span}\{u_1, u_2\} = \mathbb{R}^2$.

Clearly, $[u_1]_{(u_1,u_2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $[u_2]_{(u_1,u_2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

$\Longrightarrow \mathcal{B} = (u_1, u_2, u_3)$ is a basis of $\operatorname{span}\{u_1, u_2, u_3\}$, $[u_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[u_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $[u_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

c) $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0$:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$\Longrightarrow \mathcal{B} = (u_1, u_2, u_4)$ is a basis of $\operatorname{span}\{u_1, u_2, u_3, u_4\}$. Pivot el. in col 1, 2 and 4.

The eq. has solutions $\begin{cases} \lambda_1 = 2t \\ \lambda_2 = -t \\ \lambda_3 = t \\ \lambda_4 = 0 \end{cases}$, $t \in \mathbb{R}$. Set $t = 1$: $\begin{cases} \lambda_1 = 2 \\ \lambda_2 = -1 \\ \lambda_3 = 1 \\ \lambda_4 = 0 \end{cases}$

$\Longrightarrow 2u_1 - u_2 + u_3 = 0 \Longrightarrow u_3 = -2u_1 + u_2 + 0 \cdot u_4$

Coordinates: $[u_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[u_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $[u_3]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $[u_4]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\mathcal{B} = (u_1, u_2, u_4)$

d) Clearly, $\sum_{i=1}^{n} u_i = 0$, so $u_1, \dots, u_n$ are linearly independent,
and $u_n = \sum_{i=1}^{n-1} (-u_i) \in \operatorname{span}\{u_1, \dots, u_{n-1}\}$.

On the other hand, the equation $\lambda_1 u_1 + \dots + \lambda_{n-1} u_{n-1} = 0$ has augm. matrix

$$\left(\begin{array}{cccc|c} 1 & & & & 0 \\ -1 & 1 & & 0 & \vdots \\ & -1 & \ddots & & \vdots \\ & & \ddots & 1 & \vdots \\ 0 & & & -1 & 0 \end{array}\right) \sim \left(\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 1 & & & \vdots \\ \vdots & -1 & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & 0 & & -1 & 0 \end{array}\right) \sim \dots \sim \left(\begin{array}{cccc|c} 1 & & & & 0 \\ 0 & 1 & & & \vdots \\ & 0 & \ddots & & \vdots \\ & & \ddots & 1 & 0 \\ & & & 0 & 0 \end{array}\right),$$ so it has

unique solution $\lambda_1 = \dots = \lambda_{n-1} = 0$.
Hence the vectors $u_1, \dots, u_{n-1}$ are linearly independent, and therefore a basis of $\operatorname{span}\{u_1, \dots, u_n\}$. Setting $\underline{\underline{\mathcal{B} = (u_1, \dots, u_{n-1})}}$, we have

$$\underline{\underline{[u_i]_{\mathcal{B}} = e_i \text{ for } i = 1, 2, \dots, n-1, \text{ and } [u_n]_{\mathcal{B}} =}} \begin{pmatrix} -1 \\ -1 \\ \vdots \\ \vdots \\ -1 \end{pmatrix}$$

2.a) Set $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Eq $\lambda_1 v_1 + \lambda_2 v_2 = 0$ has augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

so v_1, v_2 are linearly independent. Since $\dim \mathbb{R}^2 = 2$, this implies that $\underline{\mathcal{B} = (v_1, v_2) \text{ is a basis of } \mathbb{R}^2}$.
By direct computation,

$$F(v_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 v_1, \quad F(v_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) v_1$$

so $[F(v_1)]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $[F(v_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and hence $\underline{\underline{[F]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}}$

b) Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $S = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

Now B is a basis of $\mathbb{R}^3$ if and only if v_1, v_2, v_3 are linearly independent, that is, if and only if $S \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Moreover, to find the matrix $[F]_B$ of F with respect to B, we need to find the coordinate vectors $[F(v_1)]_B$, $[F(v_2)]_B$, $[F(v_3)]_B$, i.e., to solve the equations

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = F(v_i) \quad \text{for } i = 1, 2, 3.$$

The augmented matrices of these eq. are $\left(S \,\middle|\, \begin{matrix} | \\ F(v_i) \\ | \end{matrix} \right)$, $i = 1, 2, 3$.

$$\begin{pmatrix} | & | & | \\ F(v_1) & F(v_2) & F(v_3) \\ | & | & | \end{pmatrix} = AS = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 3 & 4 & 5 \end{pmatrix}$$

$$\Rightarrow \left(S \,\middle|\, \begin{matrix} | & | & | \\ F(v_1) & F(v_2) & F(v_3) \\ | & | & | \end{matrix} \right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & -1 \\ 1 & 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & 1 & 3 & 4 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 4 & 5 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 3 & 3 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3/2 & 3/2 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -5/2 \\ 0 & 1 & 0 & 2 & 5/2 & 7/2 \\ 0 & 0 & 1 & 1 & 3/2 & 3/2 \end{array} \right)$$

This implies that $S \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so $\underline{\underline{B \text{ is a basis.}}}$

Moreover, reading off the three columns on the right hand side separately, we see that $[F(v_1)]_B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $[F(v_2)]_B = \frac{1}{2}\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$, $[F(v_3)]_B = \frac{1}{2}\begin{pmatrix} -5 \\ 7 \\ 3 \end{pmatrix}$, and hence

$$\underline{\underline{[F]_B = \frac{1}{2}\begin{pmatrix} 2 & -1 & -5 \\ 4 & 5 & 7 \\ 2 & 3 & 3 \end{pmatrix}}}$$

3) Set $w_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$, $w_3 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$,

The condition $[w_1]_B = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ means that $2v_1 + 0 \cdot v_2 + (-1)v_3 = w_1$,
where $B = (v_1, v_2, v_3)$ is the desired basis.

This may be written as $(2 \;\; 0 \;\; -1)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = w_1$, and similarly, the

conditions $[w_2]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $[w_3]_B = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ give $(1 \;\; 1 \;\; 1)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = w_2$, $(-1 \;\; 1 \;\; 2)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = w_3$

Collecting these equations, we get $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot w_1 + 0w_2 + 0w_3 \\ 0w_1 + 1 \cdot w_2 + 0w_3 \\ 0w_1 + 0 \cdot w_2 + 1 \cdot w_3 \end{pmatrix}$

This matrix equation can be represented by the augmented matrix

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 2 & 3 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 3 & 0 & 1 & 1 \\ 1 & -1 & -2 & 0 & 0 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & 1/2 \\ 1 & -1 & -2 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 & 1/2 \\ 1 & 0 & -1/2 & 0 & 1/2 & -1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 0 & 1/2 & -1/2 \\ 0 & 1 & 3/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

This tells us that $\begin{cases} v_1 - \frac{1}{2}v_3 = \frac{1}{2}w_2 - \frac{1}{2}w_3 \\ v_2 + \frac{3}{2}v_3 = \frac{1}{2}w_2 + \frac{1}{2}w_3 \end{cases}$, and moreover that

the vectors w_1, w_2, w_3 satisfy the eq. $w_1 - w_2 + w_3 = 0$.
We may choose any vector $v_3 \in \mathbb{R}^3$ as long as v_1, v_2, v_3 are linearly independent.

Try $v_3 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $v_1 = \frac{1}{2}v_3 + \frac{1}{2}w_2 - \frac{1}{2}w_3 = \frac{1}{2}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$,

$v_2 = -\frac{3}{2}v_3 + \frac{1}{2}w_2 + \frac{1}{2}w_3 = \frac{1}{2}\left(\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$

Check linear independence: $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$:

$$\begin{array}{c} \\ \\ 2 \quad 1 \end{array}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1/2 & 5/2 & 0 & 0 \\ -1/2 & 7/2 & 0 & 0 \end{array}\right) \sim \begin{array}{c} \\ 1/6 \\ -2 \end{array}\left(\begin{array}{ccc|c} 0 & 9 & 1 & 0 \\ 0 & 6 & 0 & 0 \\ -1/2 & 7/2 & 0 & 0 \end{array}\right) \sim \begin{array}{c} \\ -9 \quad 7 \\ \\ \end{array}\left(\begin{array}{ccc|c} 0 & 9 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -7 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$, so v_1, v_2, v_3 are linearly independent and hence a basis of $\mathbb{R}^3$.

From the earlier calculations, we know that

$[w_1]_B = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, $[w_2]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $[w_3]_B = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ with respect to this basis

3, alternative solution) We are looking for a basis $B = (v_1, v_2, v_3)$ such that the change-of-basis matrix $S = S_B = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$ satisfies

$$S\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad S\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \quad S\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \text{ i.e., } S\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ -1 & 3 & 4 \end{pmatrix} \quad (*)$$

Let $S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$. The the first row of the matrix eq. $(*)$ above gives a system $\begin{cases} 2s_{11} - s_{13} = 1 \\ s_{11} + s_{12} + s_{13} = 4 \\ -s_{11} + s_{12} + 2s_{13} = 3 \end{cases}$, the second row gives $\begin{cases} 2s_{21} - s_{23} = 1 \\ s_{21} + s_{22} + s_{23} = 3 \\ -s_{21} + s_{22} + 2s_{23} = 2 \end{cases}$

and the third row gives $\begin{cases} 2s_{31} - s_{33} = -1 \\ s_{31} + s_{32} + s_{33} = 3 \\ -s_{31} + s_{32} + 2s_{33} = 4 \end{cases}$. Combined into a single augmented

matrix: $\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 4 & 3 & 3 \\ -1 & 1 & 2 & 3 & 2 & 4 \end{array}\right) \sim \cdots \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 3/2 & 7/2 & 5/2 & 7/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Solutions: $\begin{cases} s_{11} = \frac{1}{2}t_1 + \frac{1}{2} \\ s_{12} = -\frac{3}{2}t_1 + \frac{7}{2} \\ s_{13} = t_1 \end{cases}$, $t_1 \in \mathbb{R}$; $\begin{cases} s_{21} = \frac{1}{2}t_2 + \frac{1}{2} \\ s_{22} = -\frac{3}{2}t_2 + \frac{5}{2} \\ s_{23} = t_2 \end{cases}$, $t_2 \in \mathbb{R}$.

$\begin{cases} s_{31} = \frac{1}{2}t_3 - \frac{1}{2} \\ s_{32} = -\frac{3}{2}t_3 + \frac{7}{2} \\ s_{33} = t_3 \end{cases}$, $t_3 \in \mathbb{R}$. Taking, for example, $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ gives

$$S = \begin{pmatrix} 1 & 2 & 1 \\ 1/2 & 5/2 & 0 \\ -1/2 & 7/2 & 0 \end{pmatrix}, \text{ that is, } v_1 = \begin{pmatrix} 1 \\ 1/2 \\ -1/2 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 5/2 \\ 7/2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Linear independence follows as before.

4a) Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathcal{S}_2$, $\lambda \in \mathbb{R}$.

- The zero matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ satisfies $b = 0 = 0 = c$, so $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}_2$.
- $A_1 + A_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$. Since $A_1, A_2 \in \mathcal{S}_2$, we have $b_1 = c_1$, $b_2 = c_2$ and hence $b_1 + b_2 = c_1 + c_2$. Therefore, $A_1 + A_2 \in \mathcal{S}_2$.
- $\lambda A_1 = \begin{pmatrix} \lambda a_1 & \lambda b_1 \\ \lambda c_1 & \lambda d_1 \end{pmatrix}$. Now $b_1 = c_1 \Rightarrow \lambda b_1 = \lambda c_1$, so $\lambda A_1 \in \mathcal{S}_2$.

The above proves that $\mathcal{S}_2 \subset \mathbb{R}^{2\times 2}$ is a subspace.

b) For example: $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Clearly, $M_1, M_2, M_3 \in \mathcal{S}_2$, and $\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \lambda_1 = \lambda_2 = \lambda_3 = 0$
so M_1, M_2, M_3 are linearly independent.

On the other hand $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{S}_2 \Rightarrow b = c$, so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = aM_1 + dM_2 + bM_3$
$\Rightarrow \operatorname{span}\{M_1, M_2, M_3\} = \mathcal{S}_2$

Hence $(M_1, M_2, M_3) = \mathcal{B}$ is a basis of $\mathcal{S}_2$, and $\dim \mathcal{S}_2 = 3$

c) For example: $F:\mathcal{S}_2\to\mathcal{S}_2$, $F\begin{pmatrix}a&b\\b&d\end{pmatrix}=\begin{pmatrix}a&0\\0&0\end{pmatrix}$.

• F is a linear map:

Let $A_1=\begin{pmatrix}a_1&b_1\\b_1&d_1\end{pmatrix}$, $A_2=\begin{pmatrix}a_2&b_2\\b_2&d_2\end{pmatrix}\in\mathcal{S}_2$, $\lambda\in\mathbb{R}$.

$$\cdot\ F(A_1+A_2)=F\begin{pmatrix}a_1+a_2&b_1+b_2\\b_1+b_2&d_1+d_2\end{pmatrix}=\begin{pmatrix}a_1+a_2&0\\0&0\end{pmatrix}=\begin{pmatrix}a_1&0\\0&0\end{pmatrix}+\begin{pmatrix}a_2&0\\0&0\end{pmatrix}=F(A_1)+F(A_2)$$

$$\cdot\ F(\lambda A_1)=F\begin{pmatrix}\lambda a_1&\lambda b_1\\\lambda b_1&\lambda d_1\end{pmatrix}=\begin{pmatrix}\lambda a_1&0\\0&0\end{pmatrix}=\lambda\begin{pmatrix}a_1&0\\0&0\end{pmatrix}=\lambda F(A_1)$$

• $F(M_2)=F\begin{pmatrix}0&0\\0&1\end{pmatrix}=0$, so $M_2\in\ker F\Rightarrow\ker F\neq\{0\}$

• $F(M_1)=F\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}=M_1\Rightarrow M_1\in\operatorname{im}F\Rightarrow\operatorname{im}F\neq\{0\}$.

So $F:\mathcal{S}_2\to\mathcal{S}_2$ is a <u>linear map satisfying $\ker F\neq\{0\}$</u> and <u>$\operatorname{im}F\neq\{0\}$</u>.

d)
$$\left.\begin{aligned}F(M_1)&=F\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}=M_1\\F(M_2)&=F\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\\F(M_3)&=F\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\end{aligned}\right\}\ [F]_{\mathcal{B}}=[F]_{\mathcal{B}}^{\mathcal{B}}=\begin{pmatrix}|&|&|\\ [F(M_1)]_{\mathcal{B}}&[F(M_2)]_{\mathcal{B}}&[F(M_3)]_{\mathcal{B}}\\|&|&|\end{pmatrix}$$

$$=\underline{\underline{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}}}$$

5.a) For simplicity, write $\underline{x} = (x_n)_{n=0}^{\infty} = (x_0, x_1, x_2, \dots)$, $\underline{y} = (y_n)_{n=0}^{\infty} = (y_0, y_1, y_2, \dots)$ etc.
Observe that if $\underline{x}, \underline{y} \in \ell$, then $\underline{x}+\underline{y} = (x_n+y_n)_{n=0}^{\infty} \in \ell$ and $\lambda\underline{x} = (\lambda x_n)_{n=0}^{\infty} \in \ell$.
There are seven axiom that need to be verified:

(i) Let $\underline{x}, \underline{y}, \underline{z} \in \ell$. Then $(\underline{x}+\underline{y})+\underline{z} = (x_n+y_n)_{n=0}^{\infty} + (z_n)_{n=0}^{\infty} = (x_n+y_n+z_n)_{n=0}^{\infty} =$
$= (x_n)_{n=0}^{\infty} + (y_n+z_n)_{n=0}^{\infty} = \underline{x} + (\underline{y}+\underline{z})$ (associativity)

(ii) Let $\underline{x}, \underline{y} \in \ell$. Then $\underline{x}+\underline{y} = (x_n+y_n)_{n=0}^{\infty} = (y_n+x_n)_{n=0}^{\infty} = \underline{y}+\underline{x}$ (commutativity)

(iii) Let $\underline{0} = (0)_{n=0}^{\infty} = (0,0,0,\dots) \in \ell$, and $\underline{x} \in \ell$. Then $\underline{0}+\underline{x} = (0+x_n)_{n=0}^{\infty} = (x_n)_{n=0}^{\infty} = \underline{x}$
(additive identity element)

(iv) Let $\underline{x}, \underline{y} \in \ell$, $\lambda \in \mathbb{R}$. Then
$\lambda(\underline{x}+\underline{y}) = \lambda\left((x_n)_{n=0}^{\infty} + (y_n)_{n=0}^{\infty}\right) = \lambda(x_n+y_n)_{n=0}^{\infty} = (\lambda(x_n+y_n))_{n=0}^{\infty} = (\lambda x_n + \lambda y_n)_{n=0}^{\infty}$
$= (\lambda x_n)_{n=0}^{\infty} + (\lambda y_n)_{n=0}^{\infty} = \lambda\underline{x} + \lambda\underline{y}$. (distributivity)

(v) Let $\underline{x} \in \ell$, $\lambda, \mu \in \mathbb{R}$. Then
$(\lambda+\mu)\underline{x} = ((\lambda+\mu)x_n)_{n=0}^{\infty} = (\lambda x_n + \mu x_n)_{n=0}^{\infty} = (\lambda x_n)_{n=0}^{\infty} + (\mu x_n)_{n=0}^{\infty} = \lambda\underline{x} + \mu\underline{x}$ (distributivity)

(vi) Let $\underline{x} \in \ell$, $\lambda, \mu \in \mathbb{R}$. Then $(\lambda\mu)\underline{x} = ((\lambda\mu)x_n)_{n=0}^{\infty} = (\lambda(\mu x_n))_{n=0}^{\infty} = \lambda(\mu x_n)_{n=0}^{\infty} = \lambda(\mu\underline{x})$
(mixed associativity)

(vii) Let $\underline{x} \in \ell$. Then $1 \cdot \underline{x} = (1 \cdot x_n)_{n=0}^{\infty} = (x_n)_{n=0}^{\infty} = \underline{x}$

Since all axioms are satisfied, ℓ is a vector space (linear space) with the given operations.

b) Let $\underline{x}=(x_0,x_1,x_2,\dots)$, $\underline{y}=(y_0,y_1,y_2,\dots)\in \ell$, $\lambda\in\mathbb{R}$.

• $S(\underline{x}+\underline{y})=S(x_0+y_0,x_1+y_1,\dots)=(0,x_0+y_0,x_1+y_1,\dots)=$

$=(0,x_0,x_1,\dots)+(0,y_0,y_1,\dots)=S(\underline{x})+S(\underline{y})$.

• $S(\lambda\underline{x})=S(\lambda x_0,\lambda x_1,\lambda x_2,\dots)=(0,\lambda x_0,\lambda x_1,\lambda x_2,\dots)=$

$=\lambda(0,x_0,x_1,x_2,\dots)=\lambda S(\underline{x})$

So $\underline{S:\ell\to\ell \text{ is a linear map}}$.

c) Assume that $\underline{x}=(x_0,x_1,x_2,\dots)\in\ker S$.

Then $0=S(\underline{x})=(0,x_0,x_1,x_2,\dots)$, that is: $\begin{cases}0=0\\x_0=0\\x_1=0\\x_2=0\\\vdots\\x_j=0 \text{ for all } j=0,1,2\dots\end{cases}$

This means that $\underline{x}=(0,0,0,\dots)=0$

$\Rightarrow \ker S=\{0\}$

d) A linear map $F:V\to W$ is invertible iff $\ker F=\{0\}$ and $\operatorname{im} F=W$.

But if $\underline{x}=(x_0,x_1,\dots)$, $\underline{y}=(y_0,y_1,\dots)\in\ell$ and $S(\underline{x})=\underline{y}$, then $(y_0,y_1,y_2,\dots)=(0,x_0,x_1,\dots)$, so $y_0=0$. Hence, for example, the element $\underline{y}=(1,0,0,\dots)\in\ell$ does not belong to $\operatorname{im} S$.

$\Rightarrow \operatorname{im} S\neq\ell \Rightarrow \underline{S \text{ is not invertible}}$