Linear Algebra I, autumn term 2016
Solutions to homework III

1) $U_1 = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$ is a subspace:

Assume that $x, y \in U_1$, so that $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$.

- Then $(x_1+y_1) + (x_2+y_2) + (x_3+y_3) = x_1+x_2+x_3 + y_1+y_2+y_3 = 0$, so $x+y \in U_1$.
- Moreover, if $\lambda \in \mathbb{R}$, then $\lambda x_1 + \lambda x_2 + \lambda x_3 = \lambda(x_1+x_2+x_3) = 0$, so $\lambda x \in U_1$.
- The vector $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies $0+0+0=0$, so $0 \in U_1$.

Hence, $U_1 \subset \mathbb{R}^3$ is a subspace.

- $U_2 = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$ is NOT a subspace; since $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin U_2$.

- $U_3 = \{x \in \mathbb{R}^n \mid Ax = Bx\} \subset \mathbb{R}^n$ (for given matrices $A, B \in \mathbb{R}^{n \times n}$) is a subspace:

For any $x \in \mathbb{R}^n$: $Ax = Bx \iff Ax - Bx = 0 \iff (A-B)x = 0$
$\iff x \in \ker(A-B)$.

So $U_3 = \ker(A-B) \subset \mathbb{R}^n$, and we know that the kernel of a matrix is always a subspace.

- $U_4 = \{x \in \mathbb{R}^2 \mid x_1 \leq x_2\} \subset \mathbb{R}^2$ is NOT a subspace:

Let $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$, and $\lambda = -1$. Then $u \in U_4$ (since $0 \leq 1$), but

$\lambda u = (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \notin U_4$, since $0 > -1$. Hence $U_4 \subset \mathbb{R}^2$ is not a subspace

2)
$$Av = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 3 & 6 & 5 & 1 \\ 4 & 8 & 3 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 11 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+4+11 \\ 2+8+22 \\ 3+12+55 \\ 4+16+33 \end{pmatrix} = \begin{pmatrix} 16 \\ 32 \\ 70 \\ 53 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ so } \underline{v \notin \ker A}.$$

$$Aw = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 3 & 6 & 5 & 1 \\ 4 & 8 & 3 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-4+2+1 \\ 2-8+4+2 \\ 3-12+10-1 \\ 4-16+6+6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ so } \underline{w \in \ker A}.$$

$v \in \operatorname{im} A \Leftrightarrow$ the eq. $Ax = v$ has a solution $x \in \mathbb{R}^4$
$w \in \operatorname{im} A \Leftrightarrow$ the eq. $Ax = w$ has a solution $x \in \mathbb{R}^4$

Eq. $Ax = v$ has augmented matrix $(A|v)$
Eq. $Ax = w$ has augmented matrix $(A|w)$.

The above equations can be solved together using the augm. matrix

$$(A|v\ w) = \left(\begin{array}{cccc|cc} 1 & 2 & 1 & -1 & 1 & 1 \\ 2 & 4 & 2 & -2 & 2 & -2 \\ 3 & 6 & 5 & 1 & 11 & 2 \\ 4 & 8 & 3 & -6 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 1 & 2 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 2 & 4 & 8 & -1 \\ 0 & 0 & -1 & -2 & -4 & -5 \end{array}\right)$$
(row operations: (-2), (-3), (-4) times row 1 added to rows 2, 3, 4; then rows 2 and 4 swapped)

$$\sim \left(\begin{array}{cccc|cc} 1 & 2 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -2 & -4 & -5 \\ 0 & 0 & 2 & 4 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{array}\right) \overset{(2)}{\sim} \left(\begin{array}{cccc|cc} 1 & 2 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -2 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{array}\right) \overset{(1)}{\sim} \left(\begin{array}{cccc|cc} 1 & 2 & 0 & -3 & -3 & -4 \\ 0 & 0 & -1 & -2 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{array}\right)$$

$$\overset{(-1)}{\sim} \left(\begin{array}{cccc|cc} 1 & 2 & 0 & -3 & -3 & -4 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{array}\right).$$
(the left 4×4 block is $\operatorname{rref}(A)$)

Hence, $(A|w) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -3 & -4 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$ insoluble, $\Rightarrow \underline{w \notin \operatorname{im} A}$

and $(A|v) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -3 & -3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ The eq. $Ax = v$ has a solution (actually, infinitely many)

$\Rightarrow \underline{v \in \operatorname{im} A}$

Basis of ker A: A vector $x \in \mathbb{R}^4$ belongs to ker A if and only if $Ax = 0$.

The augmented matrix of this system is $(A|0) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$,

by the earlier calculation. Hence, for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$:

$$Ax = 0 \iff \begin{cases} x_1 + 2x_2 \quad - 3x_4 = 0, \\ \qquad\qquad x_3 + 2x_4 = 0. \end{cases}$$

Set $x_2 = s$, $x_4 = t$.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s+3t \\ s \\ -2t \\ t \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

$\Rightarrow$ The vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}$ are a basis of ker A.

Basis of im A: The row-reduced echelon form of A:

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has pivot elements in the first and third columns. Therefore, the first and third columns of the matrix A form a basis of im A:

The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \\ 3 \end{pmatrix}$ are a basis of im A.

3) Set $B = \begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & t \\ 1 & 2 & 4 \\ 1 & t & (t-2)^2 \end{pmatrix}$. Then $\operatorname{im} B = \operatorname{span}\{u,v,w\} = U$

$$B = \begin{pmatrix} 1 & 2 & t \\ 1 & 2 & 4 \\ 1 & t & (t-2)^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & t \\ 0 & 0 & 4-t \\ 0 & t-2 & (t-2)^2-t \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & t \\ 0 & t-2 & t^2-5t+4 \\ 0 & 0 & 4-t \end{pmatrix} = \tilde{B}$$

Three cases:

a) If $t \neq 2$ and $t \neq 4$, then $\tilde{B}$ has rank 3, so $B \sim \tilde{B} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence u,v,w are a basis of $U = \operatorname{span}\{u,v,w\}$, and $\dim U = 3$.

b) If $t = 2$: Then $\tilde{B} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(B)$

$\operatorname{rref}(B)$ has pivot elements in the first and third columns, and hence, u,w are a basis of U. In particular, $\dim(U) = 2$.

c) If $t = 4$: Then $\tilde{B} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(B)$

$\operatorname{rref}(B)$ has pivot elements in the first and second columns, hence u,v are a basis of U. In particular, $\dim(U) = 2$.

4) Let $A = v^T = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{1\times 3}$.

Then $Ax = v^T x = v \cdot x$ for all $x \in \mathbb{R}^3$.

In particular, $V = \{x \in \mathbb{R}^3 \mid v \cdot x = 0\} = \ker A$, so $V \subset \mathbb{R}^3$ is a subspace.

$x \in V \iff Ax = 0$

$(A|0) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \end{array}\right)$. Set $x_2 = s$, $x_3 = t$: $x_1 - 2s + t = 0$.

$x_1 = 2s - t$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2s-t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R}.$$

The vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis of V.

5) <u>$V+W$ is a subspace of $\mathbb{R}^n$</u>:

· Let $x, y \in V+W$. Then there exist $v_1 \in V$, $w_1 \in W$ such that $x = v_1 + w_1$, and $v_2 \in V$, $w_2 \in W$ such that $y = v_2 + w_2$.
Since V and W are subspaces of $\mathbb{R}^n$, we know that $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$.
Hence, $x+y = (v_1+w_1)+(v_2+w_2) = \underbrace{v_1+v_2}_{\in V} + \underbrace{w_1+w_2}_{\in W} \in V+W$.

· Let $x = v_1 + w_1 \in V+W$ as above, and $\lambda \in \mathbb{R}$.
Then $\lambda x = \lambda(v_1+w_1) = \underbrace{\lambda v_1}_{\in V} + \underbrace{\lambda w_1}_{\in W} \in V+W$, where $\lambda v_1 \in V$ and $\lambda w_1 \in W$ because $V, W \subset \mathbb{R}^n$ are subspaces.

· Finally, since $0 \in V$ and $0 \in W$,
we get $0 = \underset{V}{\underset{\cap}{0}} + \underset{W}{\underset{\cap}{0}} \in V+W$.

The above proves that $V+W \subset \mathbb{R}^n$ is a subspace.

<u>$V \cap W$ is a subspace of $\mathbb{R}^n$</u>:

· Assume that $x, y \in V \cap W$. Then $x, y \in V \Longrightarrow x+y \in V$ because V is a subspace of $\mathbb{R}^n$.
Similarly, $x, y \in W \Longrightarrow x+y \in W$.
So $x+y \in V$ and $x+y \in W$, hence $x+y \in V \cap W$

· Let $x \in V \cap W$ and $\lambda \in \mathbb{R}$. $V \subset \mathbb{R}^n$ is a subsp: $x \in V, \lambda \in \mathbb{R} \Rightarrow \lambda x \in V$
$W \subset \mathbb{R}^n$ is a subspace: $x \in W$, $\lambda \in \mathbb{R} \Longrightarrow \lambda x \in W$
So $\lambda x \in V$ and $\lambda x \in W$, i.e., $\lambda x \in V \cap W$.

· $0 \in V$ and $0 \in W \Longrightarrow 0 \in V \cap W$. This proves that $V \cap W \subset \mathbb{R}^n$ is a subspace.

6) · Let $x \in \ker F$: $F(x) = 0$.

Then $GF(x) = G(F(x)) = G(0) = 0 \Rightarrow x \in \ker(GF)$

Hence $\ker F \subset \ker(GF)$

· Let $y \in \operatorname{im}(GF)$. Then there exists some $x \in \mathbb{R}^l$ such that

$y = GF(x)$, i.e., $y = G(F(x))$. Setting $w = F(x)$, we get

$y = G(w)$, so $y \in \operatorname{im} G$. This means that $\operatorname{im}(GF) \subset \operatorname{im} G$.

· Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ and $G: \mathbb{R}^2 \to \mathbb{R}^2$ both be given by the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In other words,

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \quad G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \quad (\text{and } F = G).$$

Then $\ker F = \ker G = \operatorname{span}\{e_1\}$, $\operatorname{im} F = \operatorname{im} G = \operatorname{span}\{e_1\}$.

But

$GF = F^2 = 0$, i.e., $GF(x) = 0$ for all $x \in \mathbb{R}^2$.

Hence $\ker(GF) = \mathbb{R}^2$ and $\operatorname{im}(GF) = \{0\}$, so

$\ker F \neq \ker(GF)$, and $\operatorname{im}(GF) \neq \operatorname{im} G$.