Linear Algebra I, autumn term 2016

Homework assignment 2: Solutions

1) Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1/\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 2}$. Then

$$Ax = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1/\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 \\ \frac{1}{\sqrt{2}} x_1 + 2x_2 \\ 0 \end{pmatrix} = F(x) \quad \text{for all } x \in \mathbb{R}^2.$$

Hence, F is linear, and $[F] = A$.

G is not linear, because, for example:

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda = 2$: $G(\lambda x) = G\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = G\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, while

$$\lambda G(x) = 2G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

So G does not satisfy $G(\lambda x) = \lambda G(x)$ for all $x \in \mathbb{R}^2$, and therefore is not linear.

Let $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then $H_u(x+y) = u\cdot(x+y) = u\cdot x + u\cdot y = H_u(x) + H_u(y)$

$H_u(\lambda x) = u\cdot(\lambda x) = \lambda(u\cdot x) = \lambda H_u(x)$

$\Longrightarrow$ H_u is linear.

Now, $H_u(e_i) = u\cdot e_i = u_i$, so

$$[H_u] = (u_1 \; u_2 \cdots u_n) \in \mathbb{R}^{1\times n}$$

2) Recall that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and
$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$, and that

$$[rot_\alpha] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad P_u(x) = \frac{u\cdot x}{u\cdot u}u$$

So we get

$$[rot_\alpha \circ rot_\beta] = [rot_\alpha][rot_\beta] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} = \underline{\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}}$$

$$= [rot_{\alpha+\beta}]$$

It follows that $[rot_\beta \circ rot_\alpha] = [rot_{\beta+\alpha}] = [rot_{\alpha+\beta}] = [rot_\alpha \circ rot_\beta]$

Next, $P_u(e_1) = \frac{u\cdot e_1}{u\cdot u}u = \frac{\binom{1}{2}\cdot\binom{1}{0}}{\binom{1}{2}\cdot\binom{1}{2}}\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{1^2+2^2}\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1\\2\end{pmatrix}$

and $P_u(e_2) = \frac{u\cdot e_2}{u\cdot u}u = \frac{2}{5}\begin{pmatrix}1\\2\end{pmatrix}$. Hence $[P_u] = \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}$

$$\Longrightarrow [rot_\alpha \circ P_u] = [rot_\alpha][P_u] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix} =$$

$$= \underline{\frac{1}{5}\begin{pmatrix} \cos\alpha - 2\sin\alpha & 2\cos\alpha - 4\sin\alpha \\ \sin\alpha + 2\cos\alpha & 2\sin\alpha + 4\cos\alpha \end{pmatrix}}, \quad \text{and}$$

$$[P_u \circ rot_\alpha] = \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \underline{\frac{1}{5}\begin{pmatrix} \cos\alpha + 2\sin\alpha & -\sin\alpha + 2\cos\alpha \\ 2\cos\alpha + 4\sin\alpha & -2\sin\alpha + 4\cos\alpha \end{pmatrix}}$$

To summarise:

$$[rot_\alpha \circ rot_\beta] = [rot_\beta \circ rot_\alpha] = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix},$$

$$[rot_\alpha \circ P_u] = \frac{1}{5}\begin{pmatrix} \cos\alpha - 2\sin\alpha & 2\cos\alpha - 4\sin\alpha \\ \sin\alpha + 2\cos\alpha & 2\sin\alpha + 4\cos\alpha \end{pmatrix}, \text{ and}$$

$$[P_u \circ rot_\alpha] = \frac{1}{5}\begin{pmatrix} \cos\alpha + 2\sin\alpha & 2\cos\alpha - \sin\alpha \\ 2\cos\alpha + 4\sin\alpha & 4\cos\alpha - 2\sin\alpha \end{pmatrix}$$

(note that $[rot_\alpha \circ P_u] \neq [P_u \circ rot_\alpha]$ unless $\alpha = 2\pi n$, $n \in \mathbb{Z}$)

3) Set $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $a,b,c,d \in \mathbb{R}$

Then $AB = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a-2b & c-2d \\ 2a+b & 2c+d \end{pmatrix}$

$BA = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} a+2c & -2a+c \\ b+2d & -2b+d \end{pmatrix}$

$$AB = BA \iff \begin{cases} a-2b = a+2c \\ 2a+b = b+2d \\ c-2d = -2a+c \\ 2c+d = -2b+d \end{cases} \iff \begin{cases} 2b+2c = 0 \\ 2a-2d = 0 \\ 2a-2d = 0 \\ 2b+2c = 0 \end{cases} \iff \begin{cases} a - d = 0 \\ b+c = 0 \end{cases}$$

Set $c = s$, $d = t$

$$\begin{cases} a = t \\ b = -s \\ c = s \\ d = t \end{cases} \quad s,t \in \mathbb{R}.$$

So $AB = BA \iff B = \begin{pmatrix} t & s \\ -s & t \end{pmatrix}$ for some $s,t \in \mathbb{R}$

4) F invertible $\iff$ The eq. $F\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}y_1\\y_2\end{pmatrix}$ has unique solution $\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$ for all $\begin{pmatrix}y_1\\y_2\end{pmatrix}\in\mathbb{R}^2$.

$$F\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1+2x_2\\3x_1+4x_2\end{pmatrix}$$

$$F\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}y_1\\y_2\end{pmatrix} \iff \begin{array}{c}-3\\ \\\end{array}\begin{cases}x_1+2x_2=y_1\\3x_1+4x_2=y_2\end{cases} \iff \begin{array}{c}\\ 1\end{array}\begin{cases}x_1+2x_2=y_1\\-2x_2=-3y_1+y_2\end{cases}$$

$$\iff \begin{array}{c}\\ -\frac{1}{2}\end{array}\begin{cases}x_1 \qquad =-2y_1+y_2\\-2x_2=-3y_1+y_2\end{cases} \iff \begin{cases}x_1=-2y_1+y_2\\x_2=\frac{3}{2}y_1-\frac{1}{2}y_2\end{cases} \quad \underline{\text{unique sol.}}$$

So F is invertible. The inverse of F is determined by the formula $F^{-1}(y)=x$ (where x is the solution to the eq $F(x)=y$), i.e.,

$$\underline{F^{-1}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}-2y_1+y_2\\\frac{3}{2}y_1-\frac{1}{2}y_2\end{pmatrix}}=\begin{pmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{pmatrix}$$

Similarly, the eq. $G(x)=y$ gives a linear system with augmented matrix

$$\begin{array}{c}1\\ \\ \\\end{array}\left(\begin{array}{ccc|ccc}-1&6&7&1&0&0\\1&2&3&0&1&0\\2&0&1&0&0&1\end{array}\right)\sim\begin{array}{c}2\\ \\ \\\end{array}\left(\begin{array}{ccc|ccc}-1&6&7&1&0&0\\0&8&10&1&1&0\\2&0&1&0&0&1\end{array}\right)\sim\begin{array}{c}\\ -\frac{3}{2}\\ \\\end{array}\left(\begin{array}{ccc|ccc}-1&6&7&1&0&0\\0&8&10&1&1&0\\0&12&15&2&0&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}-1&6&7&1&0&0\\0&8&10&1&1&0\\0&0&0&7/2&3/2&1\end{array}\right) \qquad \begin{cases}-x_1+6x_2+7x_3=y_1\\ \qquad 8x_2+10x_3=y_1+y_2\\ \qquad\qquad 0=\frac{7}{2}y_1+\frac{3}{2}y_2+y_3\end{cases}$$

This system does not have unique solution for all y_1,y_2,y_3, so G is not invertible.

If $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = rot_\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = rot_{-\alpha} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, so $\underline{rot_\alpha \text{ is}}$

$\underline{\text{invertible, and } rot_\alpha^{-1} = rot_{-\alpha}}$.

5a) Note that $u^T v = u \cdot v$ for all $u, v \in \mathbb{R}^n$.

Let $x \in \mathbb{R}^n$. Then $\left(\frac{1}{u \cdot u} u u^T\right) x = \frac{1}{u \cdot u} u u^T x = \frac{1}{u \cdot u} u \cdot (u \cdot x) = \frac{u \cdot x}{u \cdot u} u = P_u(x)$

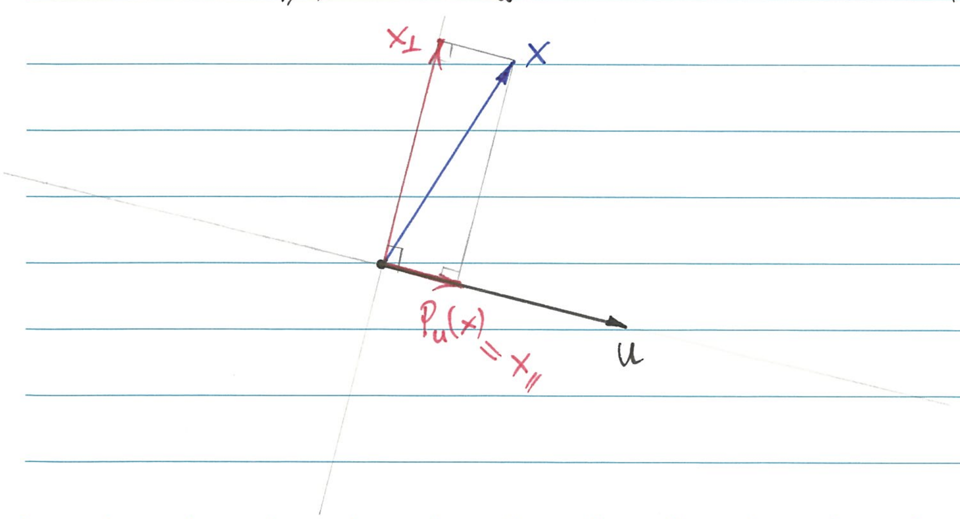
Since $\frac{1}{u \cdot u} u u^T \in \mathbb{R}^{n \times n}$, it follows that $[P_u] = \frac{1}{u \cdot u} u u^T$.

b) By the above, we get $[P_u] x = P_u(x) = \frac{u \cdot x}{u \cdot u} u$.

As $u \neq 0$, $\frac{u \cdot x}{u \cdot u} u = 0 \iff \frac{u \cdot x}{u \cdot u} = 0 \iff u \cdot x$, q.e.d.

<u>Geometric explanation</u>: Every vector x can be written as $x = x_\perp + x_\parallel$, where $x_\perp \cdot u = 0$ and $x_\parallel$ is parallel to u.

$P_u(x) = x_\parallel$, so $P_u(x) = 0 \iff x = x_\perp$, that is, x is orthogonal to u.



c) $P_u(x) = 0 \iff u \cdot x = 0$ – this equation has infinitely many solutions (unless $n=1$), so P_u is not invertible if $n \geqslant 2$.

(If $n=1$ then $P_u(x) = x$, so in this case P_u <u>is</u> invertible)