Linear Algebra I, autumn term 2016

Solutions to Homework 1

1) The augmented matrix of the system is:

$$\begin{array}{c}-2\ \ -3\\ \\ \end{array}\left(\begin{array}{ccc|c}1&3&4&5\\3&2&7&3\\2&-1&1&-4\end{array}\right)\sim\begin{array}{c}\\-1\\ \end{array}\left(\begin{array}{ccc|c}1&3&4&5\\0&-7&-5&-12\\0&-7&-7&-14\end{array}\right)\sim\begin{array}{c}\\ \\-\frac{1}{2}\end{array}\left(\begin{array}{ccc|c}1&3&4&5\\0&-7&-5&-12\\0&0&-2&-2\end{array}\right)$$

$$\sim\begin{array}{c}\\ \\-4\ \ 5\end{array}\left(\begin{array}{ccc|c}1&3&4&5\\0&-7&-5&-12\\0&0&1&1\end{array}\right)\sim\begin{array}{c}\\-\frac{1}{7}\\ \end{array}\left(\begin{array}{ccc|c}1&3&0&1\\0&-7&0&-7\\0&0&1&1\end{array}\right)\sim\begin{array}{c}\\-3\\ \end{array}\left(\begin{array}{ccc|c}1&3&0&1\\0&1&0&1\\0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&-2\\0&1&0&1\\0&0&1&1\end{array}\right)$$

The system has the unique solution $\begin{cases}x=-2\\y=1\\z=1\end{cases}$

2a) The matrices A, B and E are on row-reduced echelon form, the matrices C, D and F are not.

b) The equation $Ax=0$ is equivalent with the linear system

$$\begin{cases}x_1+2x_2 \qquad\qquad +3x_5=0\\ \qquad\qquad x_3 \qquad +2x_5=0\\ \qquad\qquad\qquad x_4+x_5=0\end{cases}$$

which is on row-reduced echelon form.

Set $\begin{cases}x_2=s\\x_5=t\end{cases}$.

$$\begin{cases}x_1=-2s-3t\\x_2=s\\x_3=-2t\\x_4=-t\\x_5=t\end{cases}$$

The solutions are:

$$x=\begin{pmatrix}-2s-3t\\s\\-2t\\-t\\t\end{pmatrix},\ \text{where } s,t\in\mathbb{R}$$

$Dy=0$ is equivalent to a linear system with augmented matrix

$$\left(D\,\middle|\,\vdots\,\right)=\left(\begin{array}{ccccc|c}1&2&0&2&0&1\\0&0&1&3&0&1\\0&0&1&4&0&1\\0&0&0&0&1&1\end{array}\right)\sim\left(\begin{array}{ccccc|c}1&2&0&2&0&1\\0&0&1&3&0&1\\0&0&0&1&0&0\\0&0&0&0&1&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccccc|c}1&2&0&0&0&1\\0&0&1&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&1\end{array}\right)$$

Set $y_2=t$. $\begin{cases}y_1=1-2t\\y_2=t\\y_3=1\\y_4=0\\y_5=1\end{cases}$

The solutions are: $y=\begin{pmatrix}1-2t\\t\\1\\0\\1\end{pmatrix}$, where $t\in\mathbb{R}$.

3) The equation $Ax=0$ is equivalent to a linear system, the augmented matrix of which is:

$$(A|0)=\left(\begin{array}{cccccc|c}2&0&-3&0&7&7&0\\-2&1&6&0&-6&-12&0\\0&1&-3&0&1&5&0\\0&-2&0&1&1&1&0\\2&1&-3&0&8&7&0\end{array}\right)\sim\left(\begin{array}{cccccc|c}2&0&-3&0&7&7&0\\0&1&3&0&1&-5&0\\0&1&-3&0&1&5&0\\0&-2&0&1&1&1&0\\0&1&0&0&1&0&0\end{array}\right)$$

$$\sim\left(\begin{array}{cccccc|c}2&0&-3&0&7&7&0\\0&1&0&0&1&0&0\\0&1&-3&0&1&5&0\\0&-2&0&1&1&1&0\\0&1&3&0&1&-5&0\end{array}\right)\sim\left(\begin{array}{cccccc|c}2&0&-3&0&7&7&0\\0&1&0&0&1&0&0\\0&0&-3&0&0&5&0\\0&0&0&1&3&1&0\\0&0&3&0&0&-5&0\end{array}\right)\sim\left(\begin{array}{cccccc|c}2&0&-3&0&7&7&0\\0&1&0&0&1&0&0\\0&0&-3&0&0&5&0\\0&0&0&1&3&1&0\\0&0&0&0&0&0&0\end{array}\right)$$

$$\sim\left(\begin{array}{cccccc|c}2&0&0&0&7&2&0\\0&1&0&0&1&0&0\\0&0&-3&0&0&5&0\\0&0&0&1&3&1&0\\0&0&0&0&0&0&0\end{array}\right)\sim\left(\begin{array}{cccccc|c}1&0&0&0&7/2&1&0\\0&1&0&0&1&0&0\\0&0&1&0&0&-5/3&0\\0&0&0&1&3&1&0\\0&0&0&0&0&0&0\end{array}\right)$$

$$\begin{cases}x_1+\frac{7}{2}x_5+x_6=0\\x_2+x_5=0\\x_3-\frac{5}{3}x_6=0\\x_4+3x_5+x_6=0\end{cases}$$

3, continued:

Set $x_5 = s$, $x_6 = t$, where s, t are arbitrary real numbers

Then $$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2}s - t \\ -s \\ \frac{5}{3}t \\ -3s - t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -\frac{7}{2} \\ -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ \frac{5}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \text{ where } s, t \in \mathbb{R},$$

are the solutions to the equation $Ax = 0$

4) Let $\begin{cases} x = \text{the weight a military horse can pull,} \\ y = \text{the weight an ordinary horse can pull,} \\ z = \text{the weight a weak horse can pull.} \end{cases}$

Then $$\begin{cases} x + y = 42 \\ 2y + z = 42 \\ x + 3z = 42 \end{cases}$$

Augmented matrix:
$$\begin{array}{c} -1 \\ \\ \\ \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 42 \\ 0 & 2 & 1 & 42 \\ 1 & 0 & 3 & 42 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 42 \\ 0 & 2 & 1 & 42 \\ 0 & -1 & 3 & 0 \end{array}\right) \sim \begin{array}{c} \\ 2 \\ \\ \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 42 \\ 0 & -1 & 3 & 0 \\ 0 & 2 & 1 & 42 \end{array}\right)$$

$$\sim \begin{array}{c} \\ \\ \frac{1}{7} \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 42 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 7 & 42 \end{array}\right) \sim \begin{array}{c} \\ \\ -3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 42 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 6 \end{array}\right) \sim \begin{array}{c} \\ 1 \\ \\ \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 42 \\ 0 & -1 & 0 & -18 \\ 0 & 0 & 1 & 6 \end{array}\right) \sim \begin{array}{c} \\ -1 \\ \\ \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 24 \\ 0 & -1 & 0 & -18 \\ 0 & 0 & 1 & 6 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & 6 \end{array}\right) \qquad \begin{cases} x = 24 \\ y = 18 \\ z = 6 \end{cases}$$

So a military horse can pull 24 dan, an ordinary horse 18 dan, and a weak horse 6 dan

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5) Let A be the matrix of F (that is, $A = [F]$).

By definition $\left.\begin{array}{l} \text{rank}(F) = \text{rank}(A) \\ \text{rank}(F) = 2 \end{array}\right\} \Longrightarrow \text{rank}(A) = 2$

$\Longrightarrow B = \text{rref}(A)$ has two pivot elements.

Hence $B = \text{rref}(A)$ has one column with no pivot element in it. (since B is a 3×3-matrix)

For all $x \in \mathbb{R}^3$, $F(x) = 0 \iff Ax = 0$. This equation is equivalent to a linear system with augm. matrix $(A|0) \sim (B|0)$. Since B has a column without a pivot element, the equation has either no solutions or infinitely many solutions.

But $x = 0$ is a solution, and hence there must exist infinitely many solutions, q.e.d.