Linear Algebra I, autumn term 2016
Solutions to the final exam, 30th January 2017

1a) Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$.

• $0 \in U$: $0 \cdot u_1 = 0$, $0 \cdot u_2 = 0$.

• Assume that $x, y \in U$. Then $(x+y) \cdot u_1 = \underbrace{x \cdot u_1}_{=0} + \underbrace{y \cdot u_1}_{=0} = 0$, as $x, y \in U$.

and $(x+y) \cdot u_2 = \overbrace{x \cdot u_2}^{=0} + \overbrace{y \cdot u_2}^{=0} = 0$. Hence, $x + y \in U$.

• Assume that $x \in U$, $\lambda \in \mathbb{R}$. Then

$$(\lambda x) \cdot u_1 = \lambda (\overbrace{x \cdot u_1}^{=0}) = 0 \text{ and } (\lambda x) \cdot u_2 = \lambda (\overbrace{x \cdot u_2}^{=0}) = 0, \text{ so } \lambda x \in U.$$

This shows that U is a subspace of $\mathbb{R}^4$

b) $v \cdot u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 1 + 1 - 3 = 0$

$v \cdot u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix} = -1 - 1 + 1 - 9 = -10 \neq 0$, therefore, $v \notin U$.

$$\left.\begin{array}{l} w \cdot u_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2 - 3 + 2 - 1 = 0 \\ w \cdot u_2 = \begin{pmatrix} 2 \\ -3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix} = -2 + 3 + 2 - 3 = 0 \end{array}\right\} \text{Hence, } w \in U.$$

The vector w is in U, the vector v is not in U

2a) Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$.

$$x \cdot u_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3 + x_4$$

$$x \cdot u_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix} = -x_1 - x_2 + x_3 + 3x_4$$

Hence, $x \in U \iff \begin{cases} x \cdot u_1 = 0 \\ x \cdot u_2 = 0 \end{cases} \iff \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 - x_2 + x_3 + 3x_4 = 0 \end{cases}$

Augmented matrix: $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right) \sim$

$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \quad \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases}$

Set $x_2 = r$, $x_4 = s$. Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -r+s \\ r \\ -2s \\ s \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}; r, s \in \mathbb{R}$

$\Rightarrow U = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$. Moreover, $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$ are linearly independent.

$\mathcal{B} = \left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right)$ is a basis of U

b) The vector v does not belong to U, so $[v]_{\mathcal{B}}$ is not defined.

For $w \in U$: Need $r, s \in \mathbb{R}$ such that $w = r\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + s\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$.

$$\begin{pmatrix}2\\-3\\2\\-1\end{pmatrix} = \begin{pmatrix}-r+s\\r\\-2s\\s\end{pmatrix} \Longrightarrow \begin{cases}-3 = r\\-1 = s\end{cases} \qquad [w]_{\mathcal{B}} = \begin{pmatrix}-3\\-1\end{pmatrix}$$

3a) Solve the equation $F(x) = 0$:

$$\left(\begin{array}{ccc|c}0&0&1&0\\0&0&-1&0\\-1&1&0&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-1&1&0&0\\0&0&-1&0\\0&0&1&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-1&1&0&0\\0&0&-1&0\\0&0&0&0\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1&-1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right) \qquad \begin{cases}x_1 - x_2 = 0\\ \quad\ x_3 = 0\end{cases} \qquad \text{Set } x_2 = t.$$

$$x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}t\\t\\0\end{pmatrix} = t\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad t \in \mathbb{R}.$$

This means that $\mathcal{B}' = \left(\begin{pmatrix}1\\1\\0\end{pmatrix}\right)$ is a basis of $\ker F$.

Since $\operatorname{rref}([F]) = \begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\end{pmatrix}$ has pivot elements in the first and third

pivot el.

columns, the first and third colums of $[F]$ constitute a basis of $\operatorname{im} F$:

$\mathcal{B}'' = \left(\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right)$ is a basis of $\operatorname{im} F$.

b) $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$ consists of three vectors.

Since $\dim \mathbb{R}^3 = 3$, it suffices to show that the vectors in $\mathcal{B}$ are linearly independent.

The equation $\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \iff \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

$\Rightarrow$ The vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ are linearly independent, and therefore $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

c) $F\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \ker F$) $\Rightarrow \left[F\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$F\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = (-1) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \left[F\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$F\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \left[F\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow [F]_{\mathcal{B}} = \begin{pmatrix} | & | & | \\ \left[F\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} & \left[F\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right]_{\mathcal{B}} & \left[F\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

4a) If $u \in \operatorname{im} F$ then, by definition, $u = F(v)$ for some $v \in V$.

$\Rightarrow F(u) = F(F(v)) = F^2(v) = 0$. Hence, $\operatorname{im} F \subset \ker F$, and, in particular, $\dim(\operatorname{im} F) \leq \dim(\ker F)$.

By the kernel-image theorem, $\dim(\ker F) + \dim(\operatorname{im} F) = \dim V = 2$, so it follows that either $\begin{cases} \dim(\ker F) = 2, \\ \dim(\operatorname{im} F) = 0, \end{cases}$ or $\begin{cases} \dim(\ker F) = 1, \\ \dim(\operatorname{im} F) = 1. \end{cases}$

But if the former is true, and $\dim(\ker F) = 2$, then

$\ker F = V$ and hence $F(v) = 0$ for all $v \in V$, i.e., $F = 0$, which contradicts the original assumptions.
Thus, $\dim(\ker F) = \dim(\operatorname{im} F) = 1$ which, together with the inclusion $\operatorname{im} F \subset \ker F$, implies that $\underline{\underline{\operatorname{im} F = \ker F}}$.

b) Assume that $[F]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for some basis $\mathcal{B}$ of V.

Then $[F^2]_{\mathcal{B}}^{\mathcal{B}} = \left([F]_{\mathcal{B}}^{\mathcal{B}}\right)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \underline{F^2 = 0}$.

Conversely, assume that $F^2 = 0$. Then, by (a), $\operatorname{im} F = \ker F \neq \{0\}$.

Let u be any non-zero vector in $\operatorname{im} F = \ker F \subset V$, and $v \in V$ a vector such that $F(v) = u$ (such a v exists, since $u \in \operatorname{im} F$).

In particular, u and v are non-zero, $F(u) = 0$ and $F(v) = u \neq 0$.

This means that u, v are linearly independent.

[Because, assume on the contrary that they were linearly dependent. Then either $u = \lambda v$ or $v = \lambda u$ for some $\lambda \in \mathbb{R}$.

- If $u = \lambda v$ then $F(u) = F(\lambda v) = \lambda F(v) = \lambda u \neq 0$, contradicting $F(u) = 0$.
- If $v = \lambda u$ then $F(v) = F(\lambda u) = \lambda F(u) = 0$, contradicting $F(v) = u \neq 0$.

Since both cases lead to a contradiction, we conclude that u, v are linearly independent.]

Since u, v are linearly independent, and $\dim V = 2$, $\mathcal{B} = (u, v)$ is a basis of V.

Moreover,

$$F(u) = 0 \Rightarrow [F(u)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$F(v) = u \Rightarrow [F(v)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{[F]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \text{ quod erat demonstrandum.}$$