Complex Analysis, autumn term 2020
Solutions to Final exam, 8th February 2021

1) $^{(y)}(-8i)^{1/3} = \exp\left(\frac{1}{3}\,{}^{(y)}\log(-8i)\right)$

$^{(y)}\log(-8i) = \ln|-8i| + \left(-\frac{\pi}{2}+2\pi k\right)i$, where $k\in\mathbb{Z}$ is such that $y \leq -\frac{\pi}{2}+2\pi k < y+2\pi$

For $y=-\pi$: $^{(-\pi)}\log(-8i) \overset{(k=0)}{=} \ln 8 - \frac{\pi}{2}i = 3\ln 2 - \frac{\pi}{2}i$

$\Longrightarrow {}^{(-\pi)}(-8i)^{1/3} = \exp\left(\frac{1}{3}\left(3\ln 2 - \frac{\pi}{2}i\right)\right) = \exp\left(\ln 2 - \frac{\pi}{6}i\right) = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$= \underline{\underline{\sqrt{3}-i}}$

For $y=0$: $^{(0)}\log(-8i) = 3\ln 2 + \frac{3\pi}{2}i$

$\Longrightarrow {}^{(0)}\log(-8i)^{1/3} = \exp\left(\frac{1}{3}\left(3\ln 2 + \frac{3\pi}{2}i\right)\right) = \exp\left(\ln 2 + \frac{\pi}{2}i\right) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \underline{\underline{2i}}$

2) Set $u(x,y) = x^3y^2$ and $v(x,y) = x^2y^3$, so that $f(x+iy) = u(x,y)+v(x,y)i$

The functions $u,v:\mathbb{R}^2\to\mathbb{R}$ are differentiable everywhere, with continuous derivatives

$\Longrightarrow f_{\mathbb{R}}:\mathbb{R}^2\to\mathbb{R}^2$, $\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}u(x,y)\\v(x,y)\end{pmatrix}$ is differentiable, with derivative

given by the matrix $\left[(Df)_{(x,y)}\right] = \begin{pmatrix}u_1'(x,y) & u_2'(x,y)\\ v_1'(x,y) & v_2'(x,y)\end{pmatrix} = \begin{pmatrix}3x^2y^2 & 2x^3y\\ 2xy^3 & 3x^2y^2\end{pmatrix}$

By the Cauchy–Riemann theorem, $f:\mathbb{C}\to\mathbb{C}$ is differentiable at $z=x+iy$

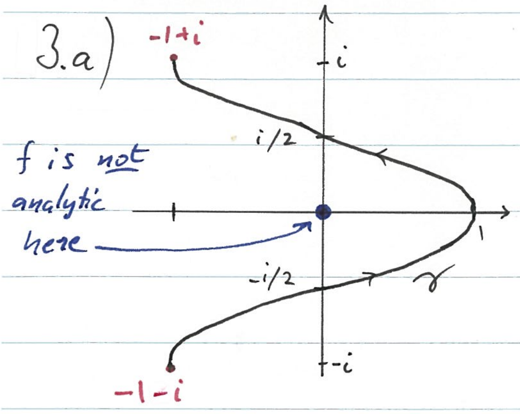
if and only if $\begin{cases}u_1'(x,y) = v_2'(x,y)\\ u_2'(x,y) = -v_1'(x,y)\end{cases} \Longleftrightarrow \begin{cases}3x^2y^2 = 3x^2y^2\\ 2x^3y = -2xy^3\end{cases}$

$\Leftrightarrow x^3y = -xy^3 \Leftrightarrow xy(x^2+y^2) = 0 \Longleftrightarrow x=0$ or $y=0$

So f is differentiable at $z=x+iy$ iff $z=x$ or $z=iy$.

Since $\left[(Df)_{(x,0)}\right] = \left[(Df)_{(0,y)}\right] = 0$, we have $f'(z)=0$ for all $z\in\mathbb{R}\cup\mathbb{R}i$.

3.a)



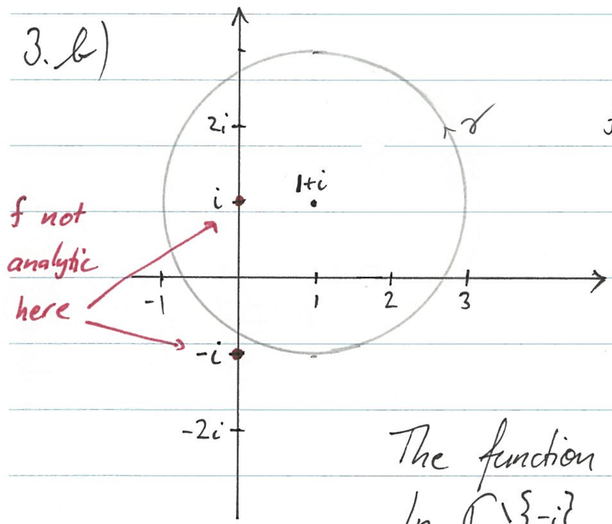
$$f(z) = \frac{1}{z}.$$

$\mathrm{Log} := {}^{(-\pi)}\log(z)$ is an antiderivative of f which is analytic along $\mathrm{im}\,\gamma$.

$$\Rightarrow \int_\gamma f = \mathrm{Log}(\gamma(1)) - \mathrm{Log}(\gamma(-1)) = \mathrm{Log}(-1+i) - \mathrm{Log}(-1-i)$$

$$= \ln\sqrt{2} + \frac{3\pi}{4}i - \left(\ln\sqrt{2} - \frac{3\pi}{4}i\right) = \underline{\underline{\frac{3\pi}{2}i}}$$

3.b)



$$f(z) = \frac{z^2}{z^2+1} = \frac{z^2+1}{z^2+1} - \frac{1}{z^2+1} = 1 - \frac{1}{z^2+1}$$

$$\frac{1}{z^2+1} = \frac{i}{2(z+i)} + \frac{-i}{2(z-i)}$$

$$\Rightarrow f(z) = 1 - \frac{i}{2(z+i)} + \frac{i}{2(z-i)}$$

The function $z \mapsto 1 - \frac{1}{2(z+i)}$ is analytic on $\mathbb{C}\setminus\{-i\}$

In $\mathbb{C}\setminus\{-i\}$, γ is homotopic to a constant curve

$\Rightarrow \int_\gamma \left(1 - \frac{i}{2(z+i)}\right) dz = 0$ by the deformation theorem.

$$\text{Now } \int_\gamma f = \int_\gamma \left(1 - \frac{i}{2(z+i)}\right) dz + \int_\gamma \frac{i}{2(z-i)}\, dz = 0 - \frac{i}{2} \underbrace{\int_\gamma \frac{1}{z-i}\, dz}_{\substack{= 2\pi i \text{ by result} \\ \text{from the course}}} = \underline{\underline{-\pi}}$$

4) <u>Claim</u>: $f^{(n)}(z) = n\exp(z) + z\exp(z)$ for all $z \in \mathbb{C}$, $n \in \mathbb{N}$

f is a product of entire functions, and thus itself entire.

$f^{(0)}(z) = f(z) = z\exp(z) = 0 \cdot \exp(z) + z\exp(z)$

Assume that, for a given $n \in \mathbb{N}$, $f^{(n)}(z) = n\exp(z) + z\exp(z)$ holds $\forall z \in \mathbb{C}$ (IH)

Then $f^{(n+1)}(z) = \frac{d}{dz} f^{(n)}(z) \overset{[IH]}{=} \frac{d}{dz}\left(n\exp(z) + z\exp(z)\right)$

$\overset{[\text{prod. rule}]}{=} n\exp(z) + \left(1 \cdot \exp(z) + z \cdot \exp(z)\right) = (n+1)\exp(z) + z\exp(z)$.

By induction, it follows that <u>$\forall z \in \mathbb{C}: f^{(n)}(z) = n\exp(z) + z\exp(z)$ holds for all $n \in \mathbb{N}$</u>.

By the power series representation theorem, as f is entire, it has a power series representation (around 0) which is <u>valid for all $z \in \mathbb{C}$</u>:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{n}{n!} z^n = \underline{\underline{\sum_{n=1}^{\infty} \frac{z^n}{(n-1)!}}}$$

5) By the power series representation theorem, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n \ \forall z \in D$.

If $f^{(n)}(w) = 0$ for all $n \in \mathbb{N}$ then $f(z) = 0$ for all $z \in D$, contradicting the assumption that f is non-zero.
Let $n \in \mathbb{N}$ be the smallest number such that $f^{(n)}(w) = 0$ (and hence $f^{(n-1)}(w) = 0$). Since $f^{(0)}(w) = f(w) = 0$, we get $n > 0$. a

It follows that $f(z) = \sum_{m=n}^{\infty} \frac{f^{(m)}(w)}{m!}(z-w)^m =$

$$= \sum_{\ell=0}^{\infty} \frac{f^{(n+\ell)}(w)}{(n+\ell)!}(z-w)^{n+\ell} = (z-w)^n \sum_{\ell=0}^{\infty} \frac{f^{(n+\ell)}(w)}{(n+\ell)!}(z-w)^\ell \quad \text{for all } z \in D.$$

In particular, the series $\sum_{\ell=0}^{\infty} \frac{f^{(n+\ell)}(w)}{(n+\ell)!}(z-w)^\ell$ converges for all $z \in D$.
By the convergence theorem for power series, a power series that converges on an open disk converges <u>normally</u> there. Thus, by Weierstrass' theorem,

$$g(z) = \sum_{\ell=0}^{\infty} \frac{f^{(n+\ell)}(w)}{(n+\ell)!}(z-w)^\ell$$

defines an analytic function $g: D \to \mathbb{C}$.
Clearly, $f(z) = (z-w)^n g(z)$ for all $z \in D$, and $g(w) = \frac{f^{(n)}(w)}{n!} \neq 0$. b

Let $\varepsilon = \frac{|g(w)|}{2}$. Since g is analytic it is in particular continuous

$$\Rightarrow \exists \delta \in]0, r[\ \forall z \in D(w;\delta) : \quad |g(z) - g(w)| < \varepsilon$$

$$|g(z)| \geq |g(w)| - |g(z) - g(w)| > |g(w)| - \varepsilon = \frac{|g(w)|}{2} > 0$$

$$\Rightarrow g(z) \neq 0.$$

That is, <u>for all $z \in D(w;\delta) \setminus \{w\}$</u>: $f(z) = \underbrace{(z-w)^n}_{\neq 0} \underbrace{g(z)}_{\neq 0} \neq 0$.