Complex Analysis, autumn term 2019

Solutions to midterm exam, 28th November 2019

1.a) $z^2-2iz+1=(z-i)^2-i^2+1=(z-i)^2+2=(z-i)^2-(\sqrt{2}i)^2=(z-i-\sqrt{2}i)(z-i+\sqrt{2}i)$

Hence, $z^2-2iz+1=0 \iff \underline{\underline{z=(1+\sqrt{2})i}}$ or $\underline{\underline{z=(1-\sqrt{2})i}}$.

b) $\cos z=\frac{1}{2}(e^{iz}+e^{-iz})$, so $\cos z=i \iff e^{iz}+e^{-iz}=2i$

$\iff e^{iz}+e^{-iz}-2i=0$

$(\forall z\in\mathbb{C}: e^{iz}\neq 0)$ $\iff (e^{iz})^2-2ie^{iz}+1=0$

Hence, by 1(a), we have

$\cos z=i \iff \underbrace{e^{iz}=(1+\sqrt{2})i}_{(1)}$ or $\underbrace{e^{iz}=(1-\sqrt{2})i}_{(2)}$

(1): $e^{iz}=(1+\sqrt{2})i=e^{\ln(1+\sqrt{2})+\frac{\pi}{2}i}$

$iz=\ln(1+\sqrt{2})+\frac{\pi}{2}i+2\pi ni$ for some $n\in\mathbb{Z}$.

$\underline{z=\frac{\pi}{2}+2\pi n-\ln(1+\sqrt{2})i, \quad n\in\mathbb{Z}}$.

(2): $e^{iz}=(1-\sqrt{2})i=e^{\ln(\sqrt{2}-1)+\frac{3\pi}{2}i}=e^{-\ln(\sqrt{2}+1)+\frac{3\pi}{2}i}$ (as $\sqrt{2}-1=(\sqrt{2}+1)^{-1}$)

$iz=-\ln(\sqrt{2}+1)+\frac{3\pi}{2}i+2\pi ni$ for some $n\in\mathbb{Z}$

$\underline{z=\frac{3\pi}{2}+2\pi n+\ln(\sqrt{2}+1)i, \quad n\in\mathbb{Z}}$.

So $\cos z=i \iff \begin{cases} z=\frac{\pi}{2}+2\pi n-\ln(1+\sqrt{2})i & \text{or} \\ z=\frac{3\pi}{2}+2\pi n+\ln(1+\sqrt{2})i & \text{for some } n\in\mathbb{Z}. \end{cases}$

2) $f(z) = {}^{(-\pi)}\log(z^{-1})$. For simplicity, write $\log = {}^{(-\pi)}\log$, and denote by $\iota : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ the function given by $\iota(z) = \frac{1}{z}$ for all $z \neq 0$. Then $f = \log\circ\iota$ and, since $\log$ is defined on $\mathbb{C}\setminus\{0\}$, we have that the domain of $f = \log\circ\iota$ is $\mathbb{C}\setminus\{0\}$.

$\log$ is analytic on $B_{-\pi} = \mathbb{C}\setminus\mathbb{R}_{\leq 0}$.
ι is analytic on $\mathbb{C}\setminus\{0\}$.
By the chain rule, f is analytic on $\iota^{-1}(B_{-\pi}) = \{z \in \mathbb{C}\setminus\{0\} \mid z^{-1} \in B_{-\pi}\}$
$= \mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

On the other hand, assume that f is differentiable at $z \in \mathbb{C}\setminus\{0\}$. Then, by the chain rule, $f\circ\iota$ is differentiable at $\iota^{-1}(z) = \iota(z) = z^{-1}$. Since $f\circ\iota = \log\circ\iota\circ\iota = \log$, this implies that $z^{-1} \in B_{-\pi} = \mathbb{C}\setminus\mathbb{R}_{\leq 0}$
$\Longrightarrow z \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

Hence, f is differentiable at z if and only if $z \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$.
For such z, $f'(z) = \frac{1}{z^{-1}}\cdot\left(-\frac{1}{z^2}\right) = -\frac{1}{z}$, by the chain rule.

The function $g(z) = \operatorname{Im} z + i|\operatorname{Re} z|$ is defined everywhere: $\mathcal{D}_g = \mathbb{C}$.
$g(x+iy) = y + i|x| = u(x,y) + iv(x,y)$, where $u(x,y) = y$, $v(x,y) = |x|$.

v is not differentiable on $\{0\}\times\mathbb{R} = \{(0,y) \in \mathbb{R}\times\mathbb{R} \mid y \in \mathbb{R}\}$
$\Rightarrow g : \mathbb{C} \to \mathbb{C}$ is not differentiable on $\{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$,
by the Cauchy-Riemann theorem.

Both u and v are C^1 on $\{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x \neq 0\} \Rightarrow g_{\mathbb{R}}$ is differentiable there.

Moreover, $\begin{cases} u'_x(x,y) = 0, & u'_y(x,y) = 1, \\ v'_x(x,y) = \operatorname{sign}(x), & v'_y(x,y) = 0. \end{cases}$

By the Cauchy-Riemann theorem, g is differentiable at $z = x+iy$ $(x \neq 0)$ if and only if $u'_x(x,y) = v'_y(x,y)$ and $v'_x(x,y) = -u'_y(x,y)$, that is,

if and only if $\operatorname{sign}(x) = -1 \iff x < 0$.
Hence, $\underline{g \text{ is analytic on } \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}}$.
On that set, $g(z) = \operatorname{Im}(z) - \operatorname{Re}(z) = -iz$, and hence $\underline{g'(z) = -i}$.

3.a) For example, $\gamma : [0,\pi] \to \mathbb{C}$, $\gamma(t) = 2e^{it}$.
Clearly, $\gamma(0) = 2$, $\gamma(\pi) = -2$, $|\gamma(t)| = |2e^{it}| = 2$ for all $t \in [0,\pi]$,
and γ is smooth: $\gamma'(t) = 2ie^{it}$ for all $t \in]0,\pi[$.

b) By definition, $\int_\gamma \bar{z}\,dz = \int_0^\pi \overline{2e^{it}} \cdot (2ie^{it})\,dt = \int_0^\pi 2e^{-it} \cdot 2ie^{it}\,dt$

$$= \int_0^\pi 4i\,dt = \underline{\underline{4\pi i}}.$$

4) Let $f_{\mathbb{R}}(x,y) = (u(x,y), v(x,y))$, where $u(x,y) = 2xy$.

Since f is analytic it satisfies the Cauchy-Riemann equations: $\begin{cases} u'_x = v'_y, \\ u'_y = -v'_x, \end{cases}$
that is, $\begin{cases} v'_y(x,y) = 2y \\ v'_x(x,y) = -2x \end{cases}$ for all $x,y \in \mathbb{R}$.

$v'_y(x,y) = 2y \implies v(x,y) = y^2 + h_1(x)$ for some differentiable function $h_1 : \mathbb{R} \to \mathbb{R}$
$v'_x(x,y) = -2x \implies v(x,y) = -x^2 + h_2(y)$ for some differentiable $h_2 : \mathbb{R} \to \mathbb{R}$.

Hence $y^2 + h_1(x) = -x^2 + h_2(y) \iff h_1(x) + x^2 = h_2(y) - y^2$
$\implies h_1(x) + x^2 = c = h_2(y) - y^2 \implies h_1(x) = -x^2 + c$ (for some $c \in \mathbb{R}$)
$\therefore f_{\mathbb{R}}(x,y) = 2xy + i(y^2 - x^2 + c)$
It follows that $\underline{\underline{f(z) = -iz^2 + ic \ (c \in \mathbb{R})}}$, which is an analytic function with $f'(z) = -2iz$.