Complex Analysis, autumn term 2019
Solutions to final exam 3rd February 2020

1. a) $\operatorname{Im}(\alpha(t)) > 0$ for all $t \in]0,1[$, $\alpha(1) = -1$, $\alpha(0) = 1$

$\Rightarrow \operatorname{im}\alpha \subset \mathbb{C} \setminus \{ai \mid a \leq 0\} =: D.$

The function ${}^{(\frac{3\pi}{2})}\log : D \to \mathbb{C}$ is an antiderivative of $z \mapsto \frac{1}{z}$ on D.

Hence, by the path independence theorem, $\underline{\underline{\int_\alpha \frac{1}{z}\,dz}} = {}^{(\frac{3\pi}{2})}\log(\alpha(1)) - {}^{(\frac{3\pi}{2})}\log(\alpha(0))$

$= {}^{(\frac{3\pi}{2})}\log(-1) - {}^{(\frac{3\pi}{2})}\log(1) = \pi i - 0 = \underline{\underline{\pi i}}$

b) The functions $z \mapsto z^2+2$ and exp are entire $\Longrightarrow z \mapsto \exp(z^2+2)$ is entire.

$z^2+2 \neq 0 \iff z \neq \pm\sqrt{2}i$

Hence, the function $z \longmapsto \dfrac{\exp(z^2+2)}{z^2+2}$ is analytic on $\mathbb{C} \setminus \{-\sqrt{2}i, \sqrt{2}i\} =: A$.

The curve γ is nullhomotopic in A

($H : [0,1]\times[0,1] \to A$, $H(s,t) = s\gamma(t)$ is a homotopy from the constant-zero curve to γ.)

Thus, by the deformation theorem, $\underline{\underline{\int_\gamma \frac{\exp(z^2+2)}{z^2+2}\,dz = 0}}$.

c) $\underline{\underline{\int_\gamma \frac{1}{|2z^3|}\,dz}} = \int_0^1 \frac{1}{|2(\gamma(t))^3|}\gamma'(t)\,dt = \int_0^1 \frac{1}{|2\cdot e^{3\cdot 2\pi i t}|}\gamma'(t)\,dt = \int_0^1 \frac{1}{2}\gamma'(t)\,dt$

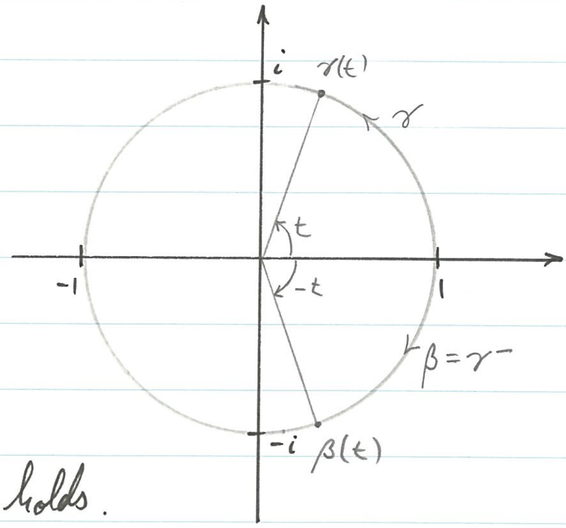
$= \frac{1}{2}\int_\gamma dz = \underline{\underline{0}}$ by Cauchy's theorem (because γ is a closed curve, the function $z \mapsto 1$ is analytic on $\mathbb{C}$, and $\mathbb{C}$ is simply connected).

d) Set $f(z) = \frac{z+2}{z^2+9}$. Then f is analytic on $\mathbb{C} \setminus \{-3i, 3i\}$, which is an open set containing the closed unit disk $\overline{D} = \overline{D(0,1)} = \{z \in \mathbb{C} \mid |z| \leq 1\}$.

By Cauchy's integral formula for derivatives, we have

$$\left.\begin{aligned} f'(0) &= \frac{1!}{2\pi i}\int_\gamma \frac{f(z)}{(z-0)^2}dz = \frac{1}{2\pi i}\int_\gamma \frac{z+2}{z^2(z^2+9)}dz \\ f'(z) &= \frac{1\cdot(z^2+9)-(z+2)\cdot 2z}{(z^2+9)^2} \implies f'(0) = \frac{9}{9^2} = \frac{1}{9} \end{aligned}\right\} \Rightarrow \int_\gamma \frac{z+2}{z^2(z^2+9)}dz = \underline{\underline{\frac{2\pi i}{9}}}$$

2.a) γ and β are parametrisations of the unit circle in $\mathbb{C}$, in the anti-clockwise and clockwise directions, respectively.
Note that $\beta(t) = \gamma(-t) = \gamma(2\pi - t)$, that is, $\beta = \gamma^-$.
Therefore, by the basic properties of the complex integral, $\underline{\underline{\int_\beta g = -\int_\gamma g}}$ holds.



b) Since $\beta(t) = e^{-it} = (e^{it})^{-1} = \gamma(t)^{-1}$, we have $\beta'(t) = -\gamma(t)^{-2}\gamma'(t)$.
Therefore,

$$\underline{\underline{\int_\beta \frac{f(z)}{z}dz}} = \int_0^{2\pi} \frac{f(\beta(t))}{\beta(t)}\beta'(t)dt = \int_0^{2\pi} \frac{f(\gamma(t)^{-1})}{\gamma(t)^{-1}}\cdot\left(-\gamma(t)^{-2}\gamma'(t)\right)dt$$

$$= -\int_0^{2\pi} f(\gamma(t)^{-1})\gamma(t)^{-1}\gamma'(t)dt = -\int_0^{2\pi}\frac{f(\gamma(t)^{-1})}{\gamma(t)}\gamma'(t)dt = \underline{\underline{-\int_\gamma \frac{f(z^{-1})}{z}dz}}$$

c) $$\underline{\underline{\int_\gamma \frac{f(z^{-1})}{z}dz}} \overset{\text{by (b)}}{=\!=} -\int_\beta \frac{f(z)}{z}dz \underset{\text{for } g(z)=\frac{f(z)}{z}}{\overset{\text{by (a)}}{=\!=}} \underline{\underline{\int_\gamma \frac{f(z)}{z}dz}}.$$

3.a) Assume that $|f(\gamma(s))| = a < C$, and set $\varepsilon = \frac{C-a}{2} \in]0, C[$

f is continuous $\Longrightarrow$ the function $A \to \mathbb{C},\ z \mapsto |f(z)|$ is continuous

$\Rightarrow \exists \delta > 0\ \forall z \in D(\gamma(s), \delta) : \big|\, |f(z)| - a \,\big| < \varepsilon$.

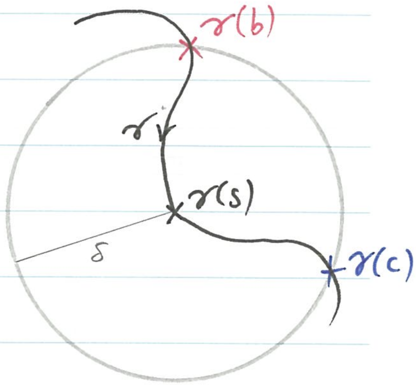
Then, by the triangle inequality, $|f(z)| \leq |a| + \varepsilon = a + \frac{C-a}{2} = \frac{C+a}{2} < C$.

Set $b = \inf\{t \in [0,s] \mid \forall u \in]t,s[: \gamma(u) \in D(\gamma(s), \delta)\}$

$c = \sup\{t \in [s,1] \mid \forall u \in]s,t[: \gamma(u) \in D(\gamma(s), \delta)\}$

By continuity, $|\gamma(b) - \gamma(s)| = \delta = |\gamma(c) - \gamma(s)|$,

so $\ell(\gamma|_{[b,c]}) \geq 2\delta > 0$.



Moreover, we have $|f(\gamma(t))| \leq \frac{C+a}{2}$ for all $t \in]b,c[$

$\Rightarrow \left| \int_{\gamma|_{[b,c]}} f \right| \leq \frac{C+a}{2} \ell(\gamma|_{[b,c]})$

Set $\gamma_1 = \gamma|_{[0,b]}$, $\gamma_2 = \gamma|_{[b,c]}$, $\gamma_3 = \gamma|_{[c,1]}$. Then $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3$, and

$$\left| \int_\gamma f \right| = \left| \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f \right| \leq \left| \int_{\gamma_1} f \right| + \left| \int_{\gamma_2} f \right| + \left| \int_{\gamma_3} f \right| \leq C\ell(\gamma_1) + \underbrace{\frac{C+a}{2}}_{<C} \ell(\gamma_2) + C\ell(\gamma_3)$$

$$< C\ell(\gamma_1) + C\ell(\gamma_2) + C\ell(\gamma_3) = C(\ell(\gamma_1) + \ell(\gamma_2) + \ell(\gamma_3)) = C\ell(\gamma)$$

That is, $\left| \int_\gamma f \right| < C\ell(\gamma)$.

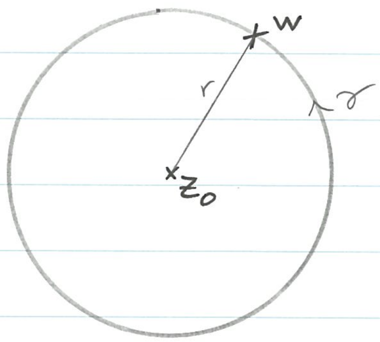
b) The set $A \subset \mathbb{C}$ is open and $z_0 \in A$ is a local maximum of $z \mapsto |f(z)|$

$\Longrightarrow \exists R>0 : \left[D(z_0,R) \subset A, \text{ and } \forall z \in D(z_0,R) : |f(z)| \leq |f(z_0)| \right]$

Set $U := D(z_0, R)$.

Assume that there exists $w \in U$ s.th. $|f(w)| < |f(z_0)| =: C$.

Let $r = |z_0 - w|$, and $\gamma : [0,2\pi] \to U$, $\gamma(t) = z_0 + re^{it}$.



As f is analytic on $U \subset A$, we have

$\int_\gamma \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$ by Cauchy's integral

theorem. Hence, $\left| \int_\gamma \frac{f(z)}{z-z_0} dz \right| = 2\pi |f(z_0)| = 2\pi C$.

On the other hand, $\left| \frac{f(z)}{z-z_0} \right| \leq \frac{C}{r}$ for all $z \in \operatorname{im} \gamma$, and $\left| \frac{f(w)}{w-z_0} \right| < \frac{C}{r}$.

Therefore, by (a): $2\pi C = \left| \int_\gamma \frac{f(z)}{z-z_0} dz \right| \overset{(a)}{<} \frac{C}{r} \ell(\gamma) = \frac{C}{r} 2\pi r = 2\pi C$.

This is a contradiction. It follows that no point $w \in U$ s.th. $|f(w)| < |f(z_0)|$ exists; that is, $z \longmapsto |f(z)|$ is constant on U.

It remains to show that f is constant on U. Let $f(x+iy) = u(x,y) + iv(x,y)$ for $u,v : A \to \mathbb{R}$. We have $|f(z)|^2 = C^2$ for all $z \in U$, that is $\underline{u^2 + v^2 = C^2}$ on U.

Differentiating this eq. with respect to x and y gives

$$\begin{cases} 2uu'_x + 2vv'_x = 0 \\ 2uu'_y + 2vv'_y = 0 \end{cases} \Longleftrightarrow 2 \underbrace{\begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix}}_{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

• If $f(w) = 0$ for some $w \in U$ then $|f(w)| = 0 \Longrightarrow |f(z)| = 0$ for all $z \in U$

$\Longrightarrow \underline{f|_U = 0 \text{ constant}}$,

· Assume that $f(z) \neq 0$ for all $z \in \mathcal{U}$. Then $\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\forall z = x+iy \in \mathcal{U}$.

By the Cauchy-Riemann equations, $A = \begin{pmatrix} u_x' & v_x' \\ u_y' & v_y' \end{pmatrix} \overset{CR}{=} \begin{pmatrix} u_x' & -u_y' \\ u_y' & u_x' \end{pmatrix}$

$\Longrightarrow \det A = (u_x')^2 + (u_y')^2 : A \longrightarrow \mathbb{R}$.

$$\left.\begin{array}{l} \text{For any } z = x+iy \in \mathcal{U}: \quad \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \neq 0 \\ \qquad A(x,y) \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = 0 \end{array}\right\} \begin{array}{l} \Longrightarrow A(x,y) \text{ is not invertible} \\ \Longrightarrow 0 = \det A(x,y) = (u_x'(x,y))^2 + (u_y'(x,y))^2 = 0 \end{array}$$

$$\Longrightarrow \begin{cases} u_x'(x,y) = 0 \\ u_y'(x,y) = 0 \end{cases} \overset{CR}{\Longrightarrow} \begin{cases} v_y'(x,y) = u_x'(x,y) = 0 \\ v_x'(x,y) = -u_y'(x,y) = 0 \end{cases}$$

Hence $f'(z) = 0$. It follows that f is constant on $\mathcal{U}$.