NAGOYA UNIVERSITY
Global 30 programme
Erik Darpö

Final exam in Basic Mathematics
Date: 3rd August 2020 *13:00 – 14:20*
Material allowed: Writing material only.

- This exam consists of seven problems, with a total score of 70 points.
- Only answers are required to the first two problems. To problems 3–7, complete solutions should be given, including motivations and clear answers to the questions asked.
- According to *Nagoya University student discipline rules* (art. 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the term.
- *Please write your name on each piece of paper.*

Only answers are required to problems 1 and 2.

1. Find the domain and the range of the function f given by the expression $f(x) = \ln(e - x^2)$. *(8 p)*

2. Draw the graph of the function f given by $f(x) = -3\sqrt{-(x+1)}$.
Draw the graph in such a way that x- and y-intercepts, domain, range, and approximate behaviour for large (positive and negative) values of x can be read off. *(8 p)*

Give full solutions to problems 3–7.

3. Write the set $\{x \in \mathbb{R} \mid x^2 \geq 4\}$ as an interval, or a union of intervals. *(8 p)*

4. Find the derivatives of the following functions.

$$f(x) = \ln(\sqrt{x} + 3), \qquad g(x) = 2^x, \qquad h(x) = e^{9x}\,(x^3 - 1)^3\,.$$

(10 p)

5. Let f be the quadratic function that has vertex $(1, 8)$ and satisfies $f(-1) = 0$. Find an explicit formula for $f(x)$ for all $x \in \mathbb{R}$. *(12 p)*

6. Solve the equation $Ax = y$, where $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.
(That is, find all vectors $x \in \mathbb{R}^3$ such that $Ax = y$.) *(12 p)*

7. Determine the equation of a straight line that contains the point $(-3, 8)$, and *precisely one point* on the curve $y = x^2$. *(12 p)*

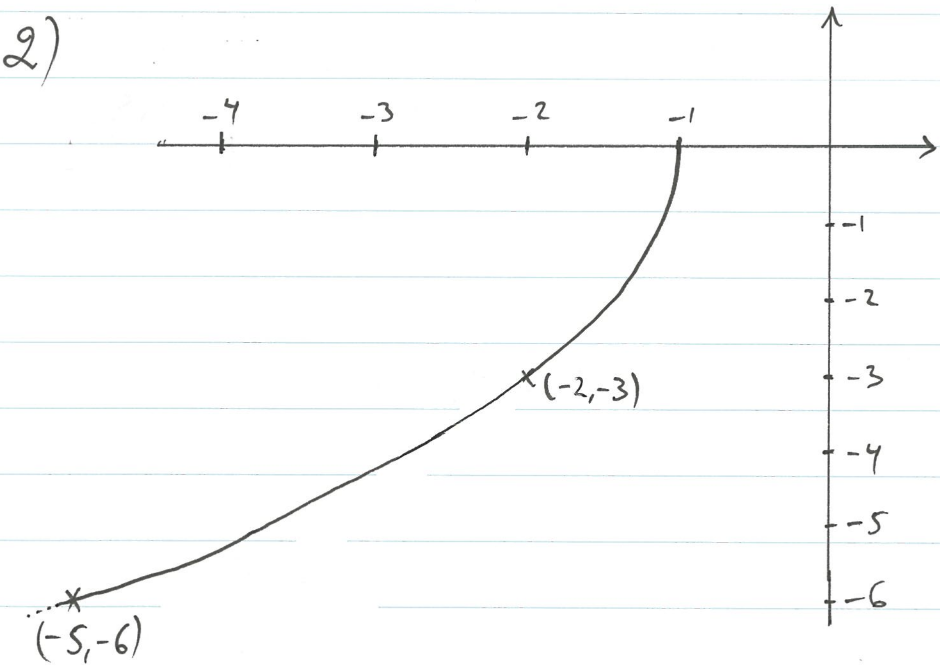
Basic Mathematics, spring term 2020
Solutions to final exam, 3rd August

1) $f(x) = \ln(e - x^2)$ is defined whenever $e - x^2 > 0$, that is, $x^2 < e$, or $-\sqrt{e} < x < \sqrt{e}$.

$$\mathcal{D}(f) =]-\sqrt{e}, \sqrt{e}[$$

$$\mathcal{R}(f) = \{ f(x) \mid -\sqrt{e} < x < \sqrt{e} \} =]-\infty, 1].$$

2)



3) $x^2 \geq 4 \iff x^2-4 \geq 0 \iff (x+2)(x-2) \geq 0$

$\iff \underline{x \leq -2 \text{ or } x \geq 2}.$

$$\{x \in \mathbb{R} \mid x^2 \geq 4\} =]-\infty;-2] \cup [2,\infty[$$

4) $f'(x) = \frac{1}{\sqrt{x}+3} \cdot \frac{1}{2\sqrt{x}} = \underline{\frac{1}{2(x+3\sqrt{x})}} \qquad (x \geq 0)$

$g(x) = 2^x = e^{\ln(2^x)} = e^{x\ln(2)}, \quad g'(x) = e^{x\ln(2)} \cdot \ln(2) = \underline{\ln(2) \cdot 2^x}.$

$h'(x) = 9e^{9x}(x^3-1)^3 + e^{9x} \cdot 3(x^3-1)^2 \cdot 3x^2$

$= 9e^{9x}(x^3-1)^2\left((x^3-1)+x^2\right) = \underline{9e^{9x}(x^3-1)^2(x^3+x^2-1)}.$

5) The function can be written on the form $f(x) = a(x-h)^2 + k$, where (h,k) is the vertex.
Here, $(h,k) = (1,8)$, so $\underline{f(x) = a(x-1)^2 + 8}$

Insert $x = -1$:

$$0 = f(-1) = a((-1)-1)^2 + 8 = 4a + 8 \implies \underline{a = -2}$$

So f is described by the formula $\underline{\underline{f(x) = -2(x-1)^2 + 8}}$.

6) The matrix-vector equation $Ax = y$ is equivalent to

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 3x_1 + 5x_2 + x_3 = 1 \\ x_1 + x_2 + 3x_3 = -1 \end{cases},$$

which has augmented matrix

$$(A|y) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & 5 & 1 & 1 \\ 1 & 1 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 1 & 1 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 0 & -1 & 4 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last matrix corresponds to the system $\begin{cases} x_1 + 7x_3 = -3, \\ x_2 - 4x_3 = 2. \end{cases}$

Set $x_3 = t$, then $\begin{cases} x_1 = -3 - 7t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$, that is, $\underline{\underline{x = \begin{pmatrix} -3-7t \\ 2+4t \\ t \end{pmatrix}}}$ $(t \in \mathbb{R})$

7) Any (non-vertical(!)) straight line containing the point $(-3,8)$ can be written on the form $y=a(x+3)+8$ for some $a\in\mathbb{R}$.

Let $f(x)=x^2$, and $g(x)=a(x+3)+8=ax+(3a+8)$ $(a\in\mathbb{R})$.

We need to find a such that the eq. $\underline{f(x)=g(x)}$ has a $\underline{\text{unique solution}}$.

$$f(x)=g(x) \iff x^2=ax+3a+8$$
$$\iff 0=x^2-ax-(3a+8)=\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}-(3a+8)$$

The equation $\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}-(3a+8)=0$ has a unique solution if and only if
$$-\frac{a^2}{4}-(3a+8)=0,$$ that is,

$$0=\frac{a^2}{4}+3a+8=4\left(a^2+12a+32\right)=4\left((a+6)^2-36+32\right)=4\left((a+6)^2-4\right)$$

$$=4\left((a+6-2)(a+6+2)\right)=4(a+4)(a+8) \iff \underline{a=-4 \text{ or } a=-8}$$

Take, say $a=-4$: $\underline{\underline{g(x)}}=-4(x+3)+8=\underline{\underline{-4x-4}}$.

The line $y=-4x-4$ has unique intersection point with the curve $y=x^2$, and contains the point $(-3,8)$

7, alternative solution)

As before, let $f(x) = x^2$, and $g(x) = a(x+3)+8$.

Notice that, if the curves $y=f(x)$ and $y=g(x)$ have a unique intersection point, then the line $y=g(x)$ is a tangent line of $y=f(x)$ at that point.

Hence, we are looking for a point $c \in \mathbb{R}$ such that $f'(c) = a$ and $f(c) = g(c)$.

$f'(x) = 2x$. The conditions above become:

$$\begin{cases} a = 2c & ① \\ c^2 = a(c+3)+8 & ② \end{cases}$$

Insert ① into ②:

$$c^2 = 2c(c+3)+8 = 2c^2+6c+8$$

$$0 = c^2+6c+8 = (c+3)^2-9+8 = (c+3)^2-1 = (c+2)(c+4).$$

$$c = -2 \quad \text{or} \quad c = -4.$$

Take $c=-2$, then $a=-4$, and $g(x) = -4(x+3)+8 = -4x-4$.

Now, we need to verify that the eq. $f(x) = g(x)$ indeed has a unique solution:

$$f(x) = g(x) \iff x^2 = -4x-4 \iff 0 = x^2+4x+4 = (x+2)^2$$

$$\iff x = -2 \quad \text{unique sol.}$$

$\therefore$ The line $y=-4x-4$ satisfies the conditions.