Basic Mathematics, spring term 2019
Solutions to midterm exam 10th June

1.a) The slope of the line $\mathcal{L}_1$ is $\frac{(-1)-1}{2-1} = -2$

Equation: $y-1 = -2(x-1)$
$\underline{\underline{y = -2x+3}}$

b) The slope of $\mathcal{L}_2$ is $-\frac{1}{(\text{slope of } \mathcal{L}_1)} = -\frac{1}{-2} = \frac{1}{2}$

The equation of $\mathcal{L}_2$ is $y = \frac{1}{2}x$.

$\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at (x_0, y_0): $\begin{cases} y_0 = -2x_0+3 \text{ and} \\ y_0 = \frac{1}{2}x_0 \end{cases}$

$\Longrightarrow \quad -2x_0+3 = \frac{1}{2}x_0$

$-4x_0+6 = x_0$
$5x_0 = 6$
$\underline{x_0 = \frac{6}{5}} \Longrightarrow y_0 = \frac{1}{2}\cdot\frac{6}{5} = \underline{\frac{3}{5}}$

The lines intersect in the point $\underline{\underline{(x_0, y_0) = \left(\frac{6}{5}, \frac{3}{5}\right)}}$.

2.a) For example, $f(x)=4x$.
Then $f(2)=4\cdot 2=8$, and
$f(-x)=4(-x)=-4x=-f(x)$ for all $x\in\mathbb{R}$,
so f is odd.

b) For example, $g(x)=8$.
Then $g(2)=8$, and $g(-x)=8=g(x)$ for all $x\in\mathbb{R}$,
so g is even.

3.a) $-1\leq 0$, so $f(-1)=\sqrt{4-(-1)}=\sqrt{5}$.

$0\leq 0$, so $f(0)=\sqrt{4-0}=\sqrt{4}=2$.

$4>0$, so $f(4)=\dfrac{4^2-4+6}{4}=\dfrac{16-4+6}{4}=\dfrac{18}{4}=\dfrac{9}{2}$.

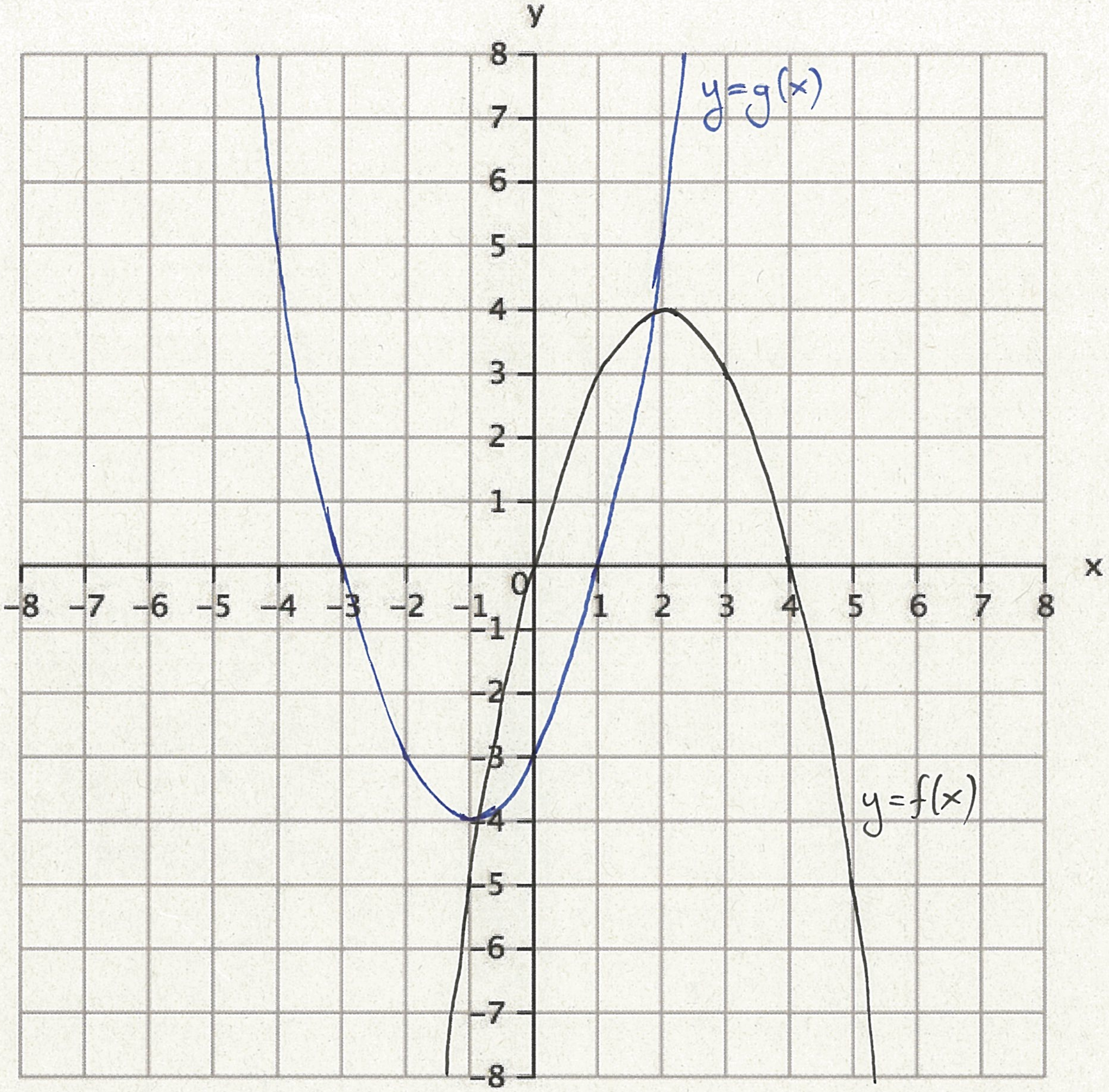
b) If $x\leq 0$: $f(x)=\sqrt{4-x}$

$f(x)=0 \iff \sqrt{4-x}=0 \iff 4-x=0 \iff x=4$.

This contradicts the assumption that $x\leq 0$.
So $f(x)\neq 0$ for all $x\leq 0$.

If $x>0$: $f(x)=\dfrac{x^2-x+6}{x}$, $f(x)=0 \overset{(x\neq 0)}{\iff} x^2-x+6=0$
$x^2-x+6=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+6=\left(x-\frac{1}{2}\right)^2+\frac{23}{4}>0$ for all $x\in\mathbb{R}$.

Hence the equation $f(x)=0$ has no solutions in $\mathbb{R}$.



y
x
y=g(x)
y=f(x)

4) $z \cdot w = (1+i)(2-i) = 2 - i + 2i - i^2 = 2 + i + 1 = \underline{\underline{3+i}}.$

$$\frac{z}{w} = \frac{1+i}{2-i} = \frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2+i+2i+i^2}{2^2+1^2} = \frac{1+3i}{5} = \underline{\underline{\frac{1}{5} + \frac{3}{5}i}}.$$

5.a) $\left.\begin{array}{l} f(x) = a(x-2)^2+4 \\ f(0) = 0 \end{array}\right\}$ $0 = a\cdot(-2)^2+4$

$0 = 4a+4$

$\underline{a = -1}$

$f(x) = -(x-2)^2+4 = -(x^2-4x+4)+4 = \underline{\underline{-x^2+4x}}.$

b) $g(x) = -f(x+3) = -\left(-(x+3-2)^2+4\right) = (x+1)^2-4$

g has vertex $(-1,-4)$, and

$g(x) = 0 \iff (x+1)^2 = 4 \iff x+1 = \pm 2 \iff \underline{x=1 \text{ or } x=-3}.$

6) $f(x) = x^5 - 3x^4 - 3x^3 + 9x^2$

$g(x) = x^3 - 3x^2 - 4x + 12$

a)

$q(x) = x^2 + 1$

$$
\begin{array}{l|l}
x^5 - 3x^4 - 3x^3 + 9x^2 & x^3 - 3x^2 - 4x + 12 \\
\underline{x^5 - 3x^4 - 4x^3 + 12x^2} & \\
\quad x^3 - 3x^2 & \\
\quad \underline{x^3 - 3x^2 - 4x + 12} & \\
\qquad 4x - 12 = r(x) &
\end{array}
$$

$$f(x) = \underbrace{(x^2+1)}_{q(x)} g(x) + \underbrace{(4x-12)}_{r(x)}$$

b) If $f(x) = g(x) = 0$ then $r(x) = \underbrace{f(x)}_{=0} - q(x)\underbrace{g(x)}_{=0} = 0$

$r(x) = 0 \iff 4x - 12 = 0 \iff x = 3.$

Check: $g(3) = 3^3 - 3 \cdot 3^2 - 4 \cdot 3 + 12 = 0$

$f(3) = q(3)g(3) + r(3) = q(3) \cdot 0 + 0 = 0$

So f and g have one common zero, $\underline{\underline{c = 3}}$.