Basic Mathematics, spring term 2019
Solutions to final exam, 29th July

1.a) $x^4+x^3-12x^2 = x^2(x^2+x-12) = x^2\left(\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-\frac{48}{4}\right) = x^2\left(\left(x+\frac{1}{2}\right)^2-\left(\frac{7}{2}\right)^2\right)$

$= x^2\left(x+\frac{1}{2}+\frac{7}{2}\right)\left(x+\frac{1}{2}-\frac{7}{2}\right) = x^2(x+4)(x-3)$

The solutions to the equation $x^4+x^3-12x^2=0$ are $\underline{\underline{x=0,\ x=-4 \text{ and } x=3}}$.

b) $2^3-3\cdot 2^2+4 = 8-12+4=0 \Rightarrow x=2$ is a solution to $x^3-3x^2+4=0$

$$
\begin{array}{l|l}
\quad x^2-x-2 & \\
\hline
x^3-3x^2+4 & x-2 \\
x^3-2x^2 & \\
\hline
\quad -x^2+4 & \\
\quad -x^2+2x & \\
\hline
\qquad -2x+4 & \\
\qquad -2x+4 & \\
\hline
\qquad\quad 0 &
\end{array}
$$

By polynomial division, $x^3-3x^2+4=(x-2)(x^2-x-2)$

$x^2-x-2 = \left(x-\frac{1}{2}\right)^2-\frac{1}{4}-\frac{8}{4} = \left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2$
$= \left(x-\frac{1}{2}-\frac{3}{2}\right)\left(x-\frac{1}{2}+\frac{3}{2}\right) = (x-2)(x+1)$

$\Rightarrow x^3-3x^2+4 = (x-2)^2(x+1)$.

The solutions of the equation $x^3-3x^2+4=0$ are $\underline{\underline{x=2 \text{ and } x=-1}}$.

c) The <u>y-intercept</u> is $f(0) = \frac{0}{4} = \underline{\underline{0}}$

The <u>x-intercepts</u> are the values of x for which $f(x)=0$,
that is, $\underline{\underline{x=-4,\ x=0 \text{ and } x=3}}$.

d) The vertical asymptotes are the lines $x=-1$ and $x=2$.

$$f(x)=\frac{x^4+x^3-12x^2}{x^3-3x^2+4}=\frac{x^2(x+4)(x-3)}{(x-2)^2(x+1)}>0 \text{ for } -4<x<-1,$$

so $f(x)\to\infty$ as $x\to -1^-$

$f(x)<0$ for $-1<x<0$, so $f(x)\to -\infty$ as $x\to -1^+$.

e)

$$\begin{array}{rl} x+4 & \\ \hline x^4+x^3-12x^2 & \big|\ x^3-3x^2+4 \\ x^4-3x^3+4x & \\ \hline 4x^3-12x^2-4x & \\ 4x^3-12x^2+16 & \\ \hline -4x-16 & \end{array}$$

By polynomial division,

$$x^4+x^3-12x^2=(x+4)(x^3-3x^2+4)-4x-16,$$

that is,

$$f(x)=(x+4)-4\,\frac{x+4}{x^3-3x^2+4}$$

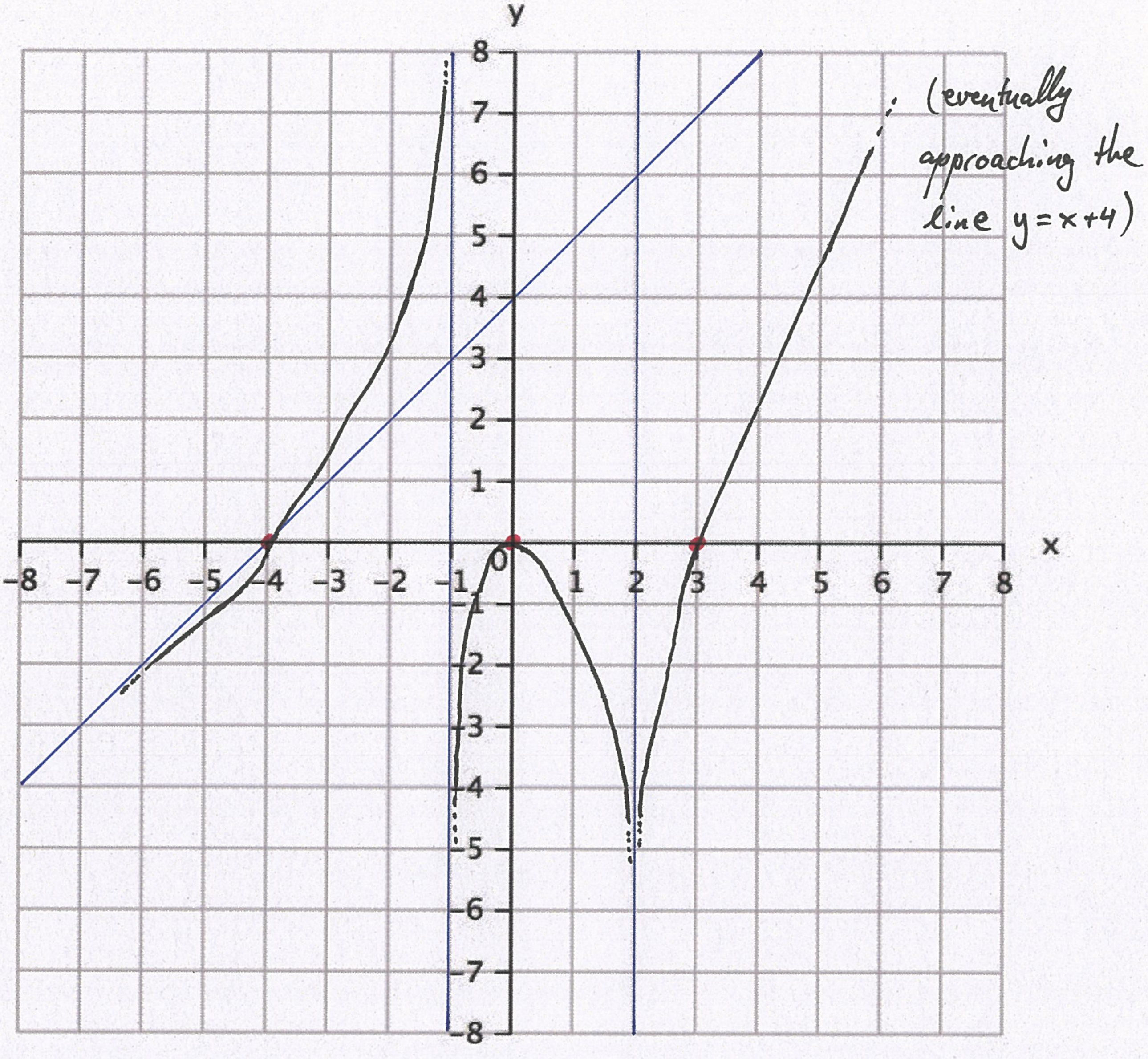
Hence $f(x)-(x+4)\to 0$ as $x\to\pm\infty$, and therefore the line $y=x+4$ is a (slant) asymptote.

For $x>-1$, $f(x)-(x+4)=-4\,\dfrac{x+4}{(x-2)^2(x+1)}<0$, meaning that

the graph $y=f(x)$ approaches the asymptote $y=x+4$ from below as $x\to\infty$.

For $x<-4$, $f(x)-(x+4)<0$, so the graph approaches the asymptote from below also as $x\to-\infty$.

f)


y
x
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-8 -7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6 7 8
(eventually approaching the line y=x+4)

2.a) Let x_t denote the number of squirrels in the forest in year Heisei t.
Then $x_{t+1} = 2x_t$, and $x_{30} = 1024$.
We are seeking t_0 such that $x_{t_0} = 2$.
For all $r \in \mathbb{N}$: $x_{t_0+r} = 2^r x_{t_0} = 2^{r+1}$
If $t_0 + r = 30$ then $1024 = x_{t_0+r} = 2^{r+1}$.
As $1024 = 2^{10}$, we get $2^{10} = 2^{r+1}$

$10 = r+1$

$r = 9$

and thus $t_0 = 30 - 9 = \underline{\underline{21}}$.

The first squirrel couple moved in nine year ago, in <u>Heisei 21</u>.

b) Let y_t be the number of chestnuts in Heisei year t.

We know that $y_{t+1} = r y_t$ or, more generally, $y_{t+u} = r^u y_t$, and also that $y_{30} = \frac{1}{2} y_{20}$.

Setting $t=20$ and $u=10$ gives $\frac{1}{2} y_{20} = y_{30} = r^{10} y_{20} \Rightarrow r^{10} = \frac{1}{2}$,

that is, $\underline{\underline{r = \frac{1}{\sqrt[10]{2}}}}$ $(\approx 0.933..)$

3) $\begin{cases} x - 3y + 2z = 4 \\ 2x + y - z = 3 \\ 3x - 2y + z = 8 \end{cases} \Leftrightarrow \begin{cases} x - 3y + 2z = 4 \\ 7y - 5z = -5 \\ 7y - 5z = -4 \end{cases}$

$\Leftrightarrow \begin{cases} x - 3y + 2z = 4 \\ 7y - 5z = -5 \\ 0 = 1 \end{cases}$ The system has no solutions.

4) $f(x) = (x^3-1)^{-1/2}$

$f'(x) = -\frac{1}{2}(x^3-1)^{-\frac{3}{2}} \cdot 3x^2 = -\dfrac{3x^2}{2\cdot\sqrt[3]{(x^3-1)^2}}$

$g'(x) = \ln x + x\frac{1}{x} = \ln x + 1$

$h(x) = x^x = e^{\ln(x^x)} = e^{x\ln x} = e^{g(x)}$ (with g as above)

$h'(x) = e^{g(x)} \cdot g'(x) = e^{x\ln x} \cdot (\ln x + 1) = x^x \cdot (\ln x + 1)$.

5. a) $x^2 - 2xy + 3y^2 = (x-y)^2 - y^2 + 3y^2 = (x-y)^2 + 2y^2 \geq 0$
since $a^2 \geq 0$ for all real numbers $a \in \mathbb{R}$.

b) Take $x = i$, $y = 0$. Then

$x^2 - 2xy + 3y^2 = i^2 + 0 + 0 = -1 < 0$.