6. The *modulus* of a complex number z is the real number $|z| = \sqrt{z\bar{z}}$.
In the following you may freely (without proof) use fundamental properties of complex numbers, such as $z_1 z_2 = z_2 z_1$, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ *and* $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$.

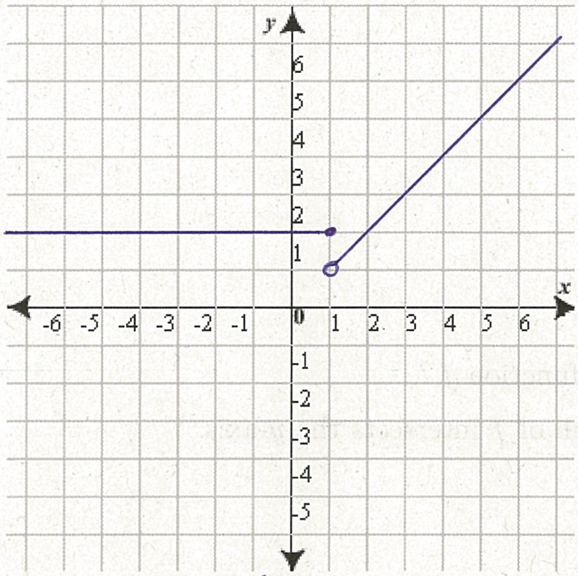
a) Show that $|z| \geq 0$, and that $|z| = 0$ if and only if $z = 0$.

b) Show that $|zw|^2 = |z|^2 |w|^2$ for all $z, w \in \mathbb{C}$.

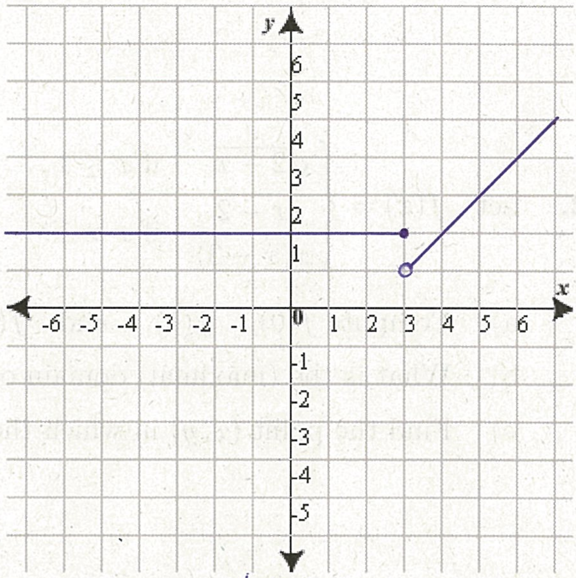
c) Show that $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$. *Hint: Use (a) and (b).*

d) Show that if $zw = 0$ then either $z = 0$ or $w = 0$. *Hint: Use (a) and (c).*

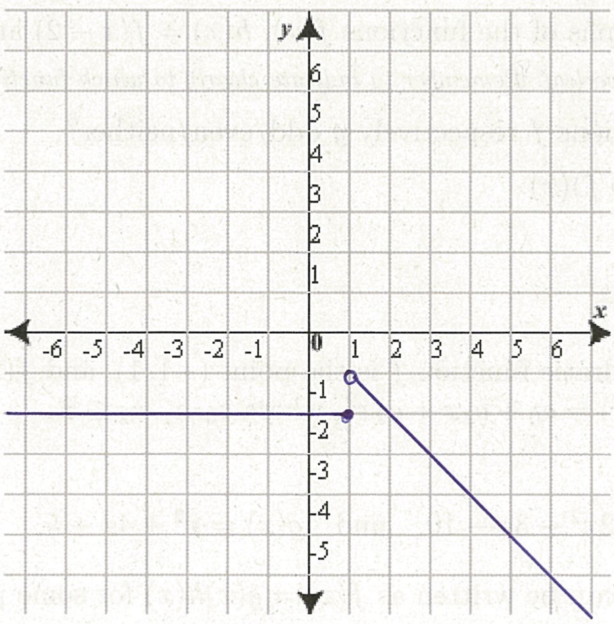
(6)



$y=f(x)$



$y=h(x)$



$y=i(x)$

Basic Mathematics:

Solutions to midterm exam, 30th May 2016

1. a) Let $y = lx + m$ be the eq. of $\mathcal{L}_1$.

Then $l = \frac{2-(-1)}{1-(-1)} = \frac{3}{2}$. Hence $y = \frac{3}{2}x + m$

Insert a point: $2 = \frac{3}{2} \cdot 1 + m$

$$2 = \frac{3}{2} + m$$

$$m = 2 - \frac{3}{2} = \frac{1}{2}.$$

The eq. of $\mathcal{L}_1$ is $\underline{\underline{y = \frac{3}{2}x + \frac{1}{2}}}$

b) Since the lines are parallel, they have the same slope.

$\mathcal{L}_2$: $y = \frac{3}{2}x + m_2$. Insert point $(2,6)$:

$$6 = \frac{3}{2} \cdot 2 + m_2$$

$m_2 = 6 - \frac{3}{2} \cdot 2 = 6 - 3 = 3$. So $\mathcal{L}_2$: $\underline{\underline{y = \frac{3}{2}x + 3}}$

c) The eq. of $\mathcal{L}_3$ is $y = l_3 x + m_3$.

Since $\mathcal{L}_1$ and $\mathcal{L}_3$ are perpendicular $l_3 \cdot l = -1$

$$l_3 \cdot \frac{3}{2} = -1 \implies l_3 = -\frac{2}{3}$$

Eq. $\mathcal{L}_3$: $y = -\frac{2}{3}x + m_3$. Insert point $(1,2)$:

$$2 = -\frac{2}{3} \cdot 1 + m_3$$

$m_3 = 2 + \frac{2}{3} = \frac{8}{3}$ So $\mathcal{L}_3$: $\underline{\underline{y = -\frac{2}{3}x + \frac{8}{3}}}$

2. $f(x) = \begin{cases} \sqrt{2-x} & \text{if } x \le 1, \\ \dfrac{x-2}{x(x-3)} & \text{if } x > 1. \end{cases}$

a) $f(0) = \sqrt{2-0} = \underline{\underline{\sqrt{2}}}$

$f(1) = \sqrt{2-1} = \sqrt{1} = \underline{\underline{1}}$

$f\left(\frac{3}{2}\right) = \dfrac{\frac{3}{2}-2}{\frac{3}{2}\left(\frac{3}{2}-3\right)} = \dfrac{-\frac{1}{2}}{\frac{3}{2}\cdot\left(-\frac{3}{2}\right)} = \dfrac{\frac{1}{2}}{\frac{9}{4}} = \underline{\underline{\dfrac{2}{9}}}$

b) We first seek the points where f is <u>not defined</u>:

- $\sqrt{2-x}$ is undefined when $2-x<0$

  Hence f is undefined when $2-x<0$ and $x \le 1$

  $2 > x$ and $x \le 1$

  <u>No such points</u> exist.

- $\dfrac{x-2}{x(x-3)}$ is undefined when $x=0$ or $x-3=0$

  $x=3$

  So f is undefined when $x>1$ and $(x=0$ or $x=3)$,

  that is, when $\underline{x=3}$.

The function f is defined for all $x \in \mathbb{R}$ except $x=3$.

That is, the domain is $\underline{\underline{\mathcal{D}(f) = \mathbb{R} \setminus \{3\}}}$

c) The graph of f intersects the y-axis in the point $(0, f(0))$,

that is, $\underline{\underline{(0, \sqrt{2})}}$

3. $f(x)=\begin{cases} 2 & \text{if } x\le 1, \\ x & \text{if } x>1, \end{cases} \quad g(x)=x^2+1$

b) • $f(3)=3$, $f(-3)=2$

Since $f(-3)\neq f(3)$ and $f(-3)\neq -f(3)$, the function f is neither odd nor even.

• For all $x\in\mathbb{R}$: $g(-x)=(-x)^2+1=x^2+1=g(x)$, so g is even.

c) If $x\le 1$: $(g\circ f)(x)=g(f(x))=g(2)=2^2+1=5$.

If $x>1$: $(g\circ f)(x)=g(f(x))=g(x)=x^2+1$.

Hence, $(g\circ f)(x)=\begin{cases} 5 & \text{if } x\le 1, \\ x^2+1 & \text{if } x>1. \end{cases}$

4) Every quadratic function can be written on the form $f(x)=\lambda(x-a)^2+b$, where $\lambda,a,b\in\mathbb{R}$ and (a,b) is the vertex of f.

$\Rightarrow f(x)=\lambda(x+1)^2+1$. Since $f(2)=19$, we have

$19=\lambda(2+1)^2+1=9\lambda+1$

$9\cdot\lambda=18$

$\lambda=2$. So $f(x)=2(x+1)^2+1=2\cdot(x^2+2x+1)+1=2x^2+4x+3$

$f(x)=2x^2+4x+3$

5. $f(x) = x^3 + 2x^2 - 3x - 10$

$g(x) = x^2 + 4x + 5$

a) Polynomial division:

$$\begin{array}{l|l} \multicolumn{1}{l}{x-2} & \\ \hline x^3+2x^2-3x-10 & \boxed{x^2+4x+5} \\ -(x^3+4x^2+5x) & \\ \hline \quad -2x^2-8x-10 & \\ -(-2x^2-8x-10) & \\ \hline \qquad\qquad 0 & \end{array}$$

So $\underline{\underline{f(x) = g(x) \cdot (x-2)}}$

b) $f(x) = 0 \iff g(x) = 0 \quad \text{or} \quad x - 2 = 0$

$g(x) = x^2 + 4x + 5 = (x+2)^2 - 4 + 5 = (x+2)^2 + 1$

$= (x+2)^2 - i^2 = (x+2+i)(x+2-i)$

Now $g(x) = 0 \iff x + 2 + i = 0 \quad \text{or} \quad x + 2 - i = 0$

$\underline{x = -2 - i} \quad \text{or} \quad \underline{x = -2 + i}$

Hence, $f(x) = 0 \iff \underline{\underline{x = 2 \text{ or } x = -2 - i \text{ or } x = -2 + i}}$

6. a) Let $z = a + bi$, $a, b \in \mathbb{R}$.

Then $|z| = \sqrt{z\bar{z}} = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2 - abi + abi - (bi)^2} = \sqrt{a^2+b^2}$.

· If $z = 0$: Then $|z| = \sqrt{0 \cdot \bar{0}} = \sqrt{0} = 0$

· If $|z| = 0$: Then $\sqrt{a^2+b^2} = 0 \Rightarrow a^2+b^2 = 0 \xRightarrow{\text{(because } a,b\in\mathbb{R})} a = b = 0$

$\Rightarrow z = 0$.

b) Let $z, w \in \mathbb{C}$.

$|z|^2 = \sqrt{z\bar{z}}^2 = z\bar{z}$, $|w|^2 = w\bar{w}$, so $|z|^2|w|^2 = z\bar{z} \cdot w\bar{w}$

$|zw|^2 = \sqrt{(zw)(\overline{zw})}^2 = zw\,\overline{zw} = zw\bar{z}\bar{w} = z\bar{z} \cdot w\bar{w} = |z|^2|w|^2$

c) Since $|zw|^2 = |z|^2 \cdot |w|^2 = (|z| \cdot |w|)^2$, (by (b))

and $|zw|$ and $|z| \cdot |w|$ are non-negative (≥ 0), (by (a))

it follows that

$|zw| = \sqrt{|zw|^2} = \sqrt{(|z| \cdot |w|)^2} = |z| \cdot |w|$

d) If $zw = 0$ then $|zw| = 0$ (by (a))

By (c), $|z| \cdot |w| = |zw| = 0$, so $|z| = 0$ or $|w| = 0$

$\xRightarrow{\text{by (a)}}$ $z = 0$ or $w = 0$

Q.E.D.