Name:

Basic Mathematics - Midterm examination

Duration: 90 minutes.

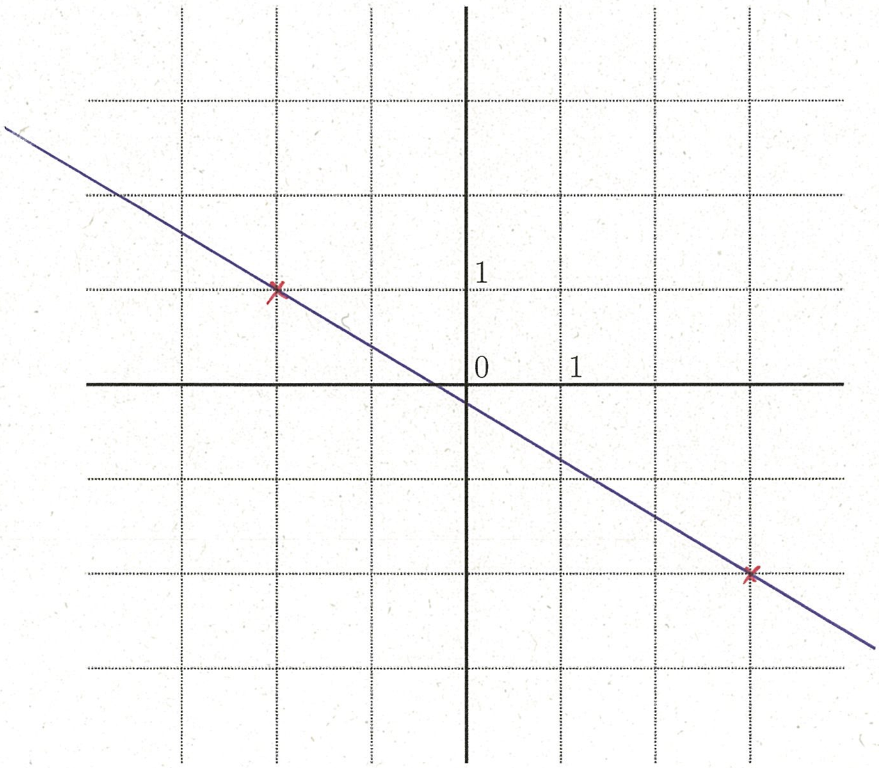
Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1 : (5)

1. Draw the line L passing through the points $(-2, 1)$ and $(3, -2)$.



2. Compute the equation of L.

$y = lx + m$

$l = \frac{1-(-2)}{-2-3} = -\frac{3}{5}$

$y = -\frac{3}{5}x + m$

$1 = -\frac{3}{5}\cdot(-2) + m$

$1 = \frac{6}{5} + m$

$m = \frac{5}{5} - \frac{6}{5} = -\frac{1}{5}$

The eq. is $\underline{y = -\frac{3}{5}x - \frac{1}{5}}$

3. Compute the equation of the line parallel to L passing through $(3,7)$.

The eq. of this line is $y=-\frac{3}{5}x+m'$, and $(3,7)$ satisfies this eqation.

$7=-\frac{3}{5}\cdot 3+m'$

$m'=7+\frac{9}{5}=\frac{35}{5}+\frac{9}{5}=\frac{44}{5}$

So the eq. is $\underline{y=-\frac{3}{5}x+\frac{44}{5}}$

4. Compute the equation of the line perpendicular to L passing through $(-4,3)$.

If two lines $L_1: y=l_1x+m_1$ and $L_2: y=l_2x+m_2$ are perpendicular, then $l_1 \cdot l_2=-1$. In our case, $l_1=-\frac{3}{5}$, so $l_2=-\frac{1}{l_1}=\frac{5}{3}$.

$y=\frac{5}{3}x+m_2$

$3=\frac{5}{3}\cdot(-4)+m_2$

$m_2=\frac{9}{3}+\frac{20}{3}=\frac{29}{3}$

The eq is $\underline{y=\frac{5}{3}x+\frac{29}{3}}$

Problem 2: (8) We consider the function f defined by:

$$f(x)=\begin{cases}(x+2)^2-1 & \text{if } x<-1;\\ \dfrac{6}{(x+3)(x-1)} & \text{if } x\geqslant -1.\end{cases}$$

1. Compute:

$f(x)=(x+2)^2-1$

- $f(-4)=(-4+2)^2-1=3$
- $f(-3)=(-3+2)^2-1=0$
- $f(-2)=(-2+2)^2-1=-1$

$f(x)=\frac{6}{(x+3)(x-1)}$

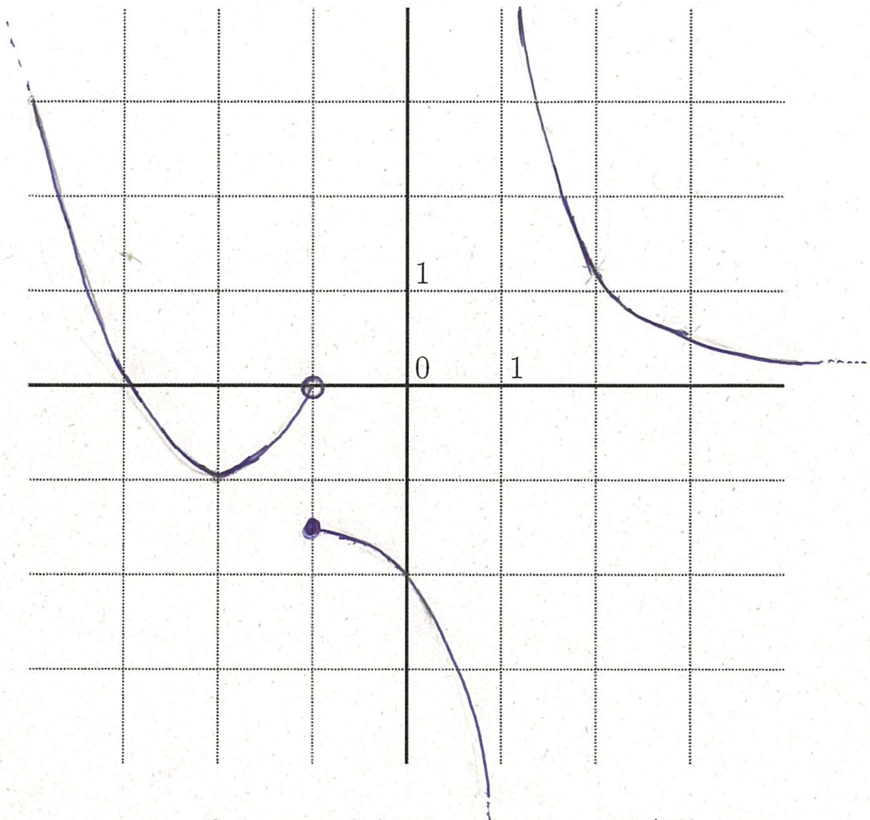
- $f(-1)=6/((-1+3)\cdot(-1-1))=6/(-4)=-\frac{3}{2}$
- $f(0)=-2$
- $f(1/2)=-24/7$
- $f(3/2)=24/9$
- $f(2)=6/5$
- $f(3)=1/2$

2. What is the maximal domain of f?

The formula for f is defined for all $x\in\mathbb{R}$, except when $x\geqslant -1$ <u>and</u> ($x=-3$ or $x=1$), that is, for all $x\in\mathbb{R}$ except $x=1$.

So the domain is $\underline{\mathcal{D}(f)=\mathbb{R}\setminus\{1\}}$

3. Draw the graph of f.



4. What is the average rate of change of f between -2 and $3/2$?

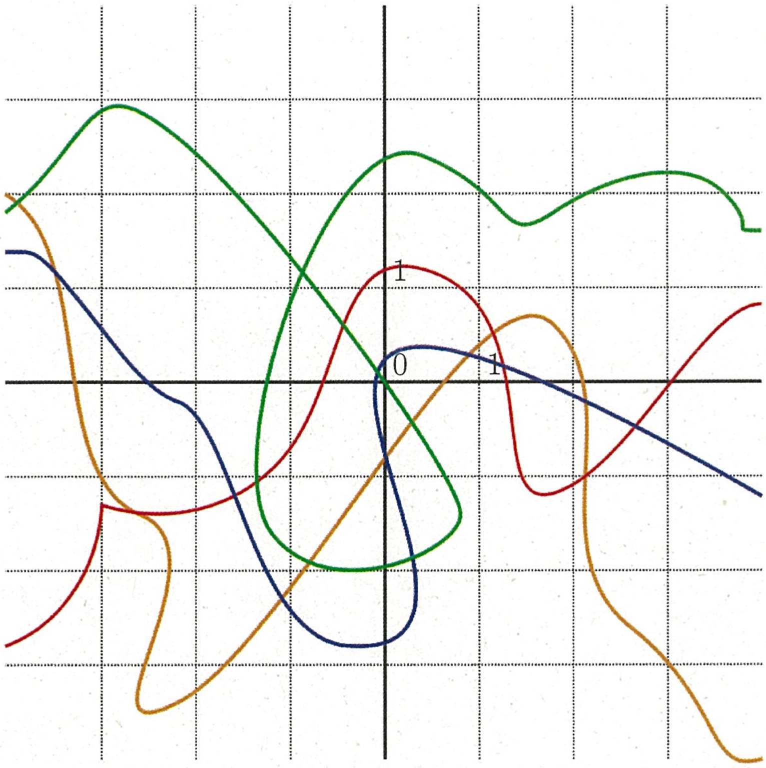
The average rate of change is

$$\frac{f(\frac{3}{2})-f(-2)}{\frac{3}{2}-(-2)} = \frac{\frac{24}{9}-(-1)}{\frac{3}{2}+2} = \frac{\frac{24}{9}+\frac{9}{9}}{\frac{3}{2}+\frac{4}{2}} = \frac{2\cdot 33}{9\cdot 7} =$$

$$= \frac{2\cdot 11}{3\cdot 7} = \underline{\underline{\frac{22}{21}}}$$

Note: We did not talk about the average rate of change yet in this course, we will cover it later.

Problem 3 : **(4)** We consider the following curves:



For each of them, tell if it is the graph of a function and justify your answer:

1. Red

   Yes

2. Green

   No - function value at (e.g.,) $x=0$ undefined.

3. Blue

   No - undefined at $x=0$

4. Orange

   No - undefined near $x=-\frac{5}{2}$.

At the indicated x-values above, the corresponding vertical line will intersect the curve at multiple points, which means that the graph cannot be that of a function.

Problem 4: (10) We consider the functions from $\mathbb{R}$ to $\mathbb{R}$ defined by:

$$f(x) = 2x - 3$$

$$g(x) = x^2 - 3x + 2$$

1. Compute $f \circ g$.

$$(f\circ g)(x) = f(g(x)) = 2(x^2-3x+2) - 3 = (2x^2-6x+4) - 3$$
$$= \underline{\underline{2x^2-6x+1}}$$

2. Compute $g \circ f$.

$$(g\circ f)(x) = g(f(x)) = (2x-3)^2 - 3(2x-3) + 2 =$$
$$= 4x^2-12x+9-6x+9+2 = \underline{\underline{4x^2-18x+20}}$$

3. Compute $f - 3g$.

$$(f-3g)(x) = f(x) - 3g(x) = 2x-3-3\cdot(x^2-3x+2) =$$
$$= 2x-3-3x^2+9x-6 = \underline{\underline{-3x^2+11x-9}}$$

4. Compute f^{-1}.

Set $y = f(x)$. $y = 2x-3$

$2x = y+3$

$x = \frac{y+3}{2}$

Now $x = f^{-1}(y)$, i.e.,

$\underline{\underline{f^{-1}(y) = \frac{y+3}{2}}}$

5. Solve the equation $f(x) = 0$.

$f(x) = 0$

$2x-3 = 0$

$2x = 3$

$\underline{\underline{x = \frac{3}{2}}}$

6. Solve the equation $g(x) = 0$.

$$g(x) = x^2-3x+2 = \left(x-\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2 + 2 = \left(x-\tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{8}{4} = \left(x-\tfrac{3}{2}\right)^2 - \tfrac{1}{4} =$$
$$= \left(x-\tfrac{3}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 = \left(x-\tfrac{3}{2}-\tfrac{1}{2}\right)\left(x-\tfrac{3}{2}+\tfrac{1}{2}\right) = (x-2)(x-1).$$

Hence $g(x) = 0$

$\iff (x-2)(x-1) = 0$

$\iff \underline{\underline{x = 1 \text{ or } x = 2}}$

OBS $\longrightarrow$ 7. What is the vertex of ~~f?~~ g?

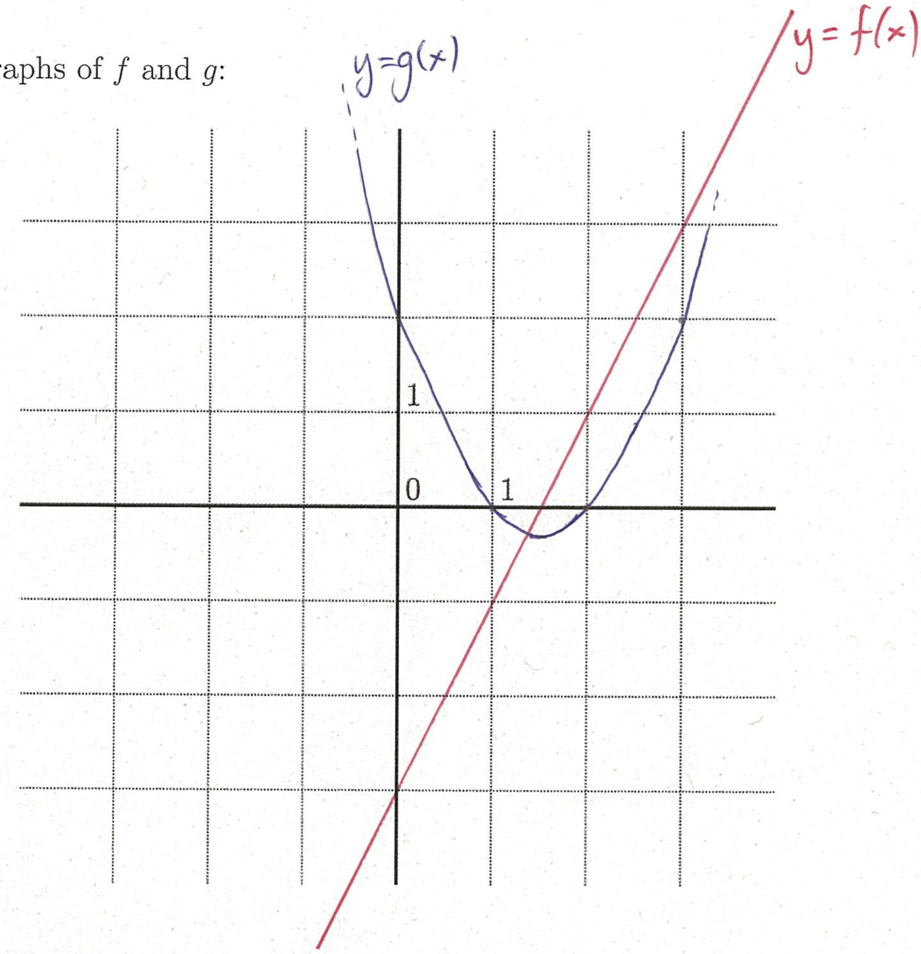
By (6), we know that $g(x) = \left(x-\frac{3}{2}\right)^2 - \frac{1}{4}$.

Thus the minimal value of $g(x)$ is $-\frac{1}{4}$,

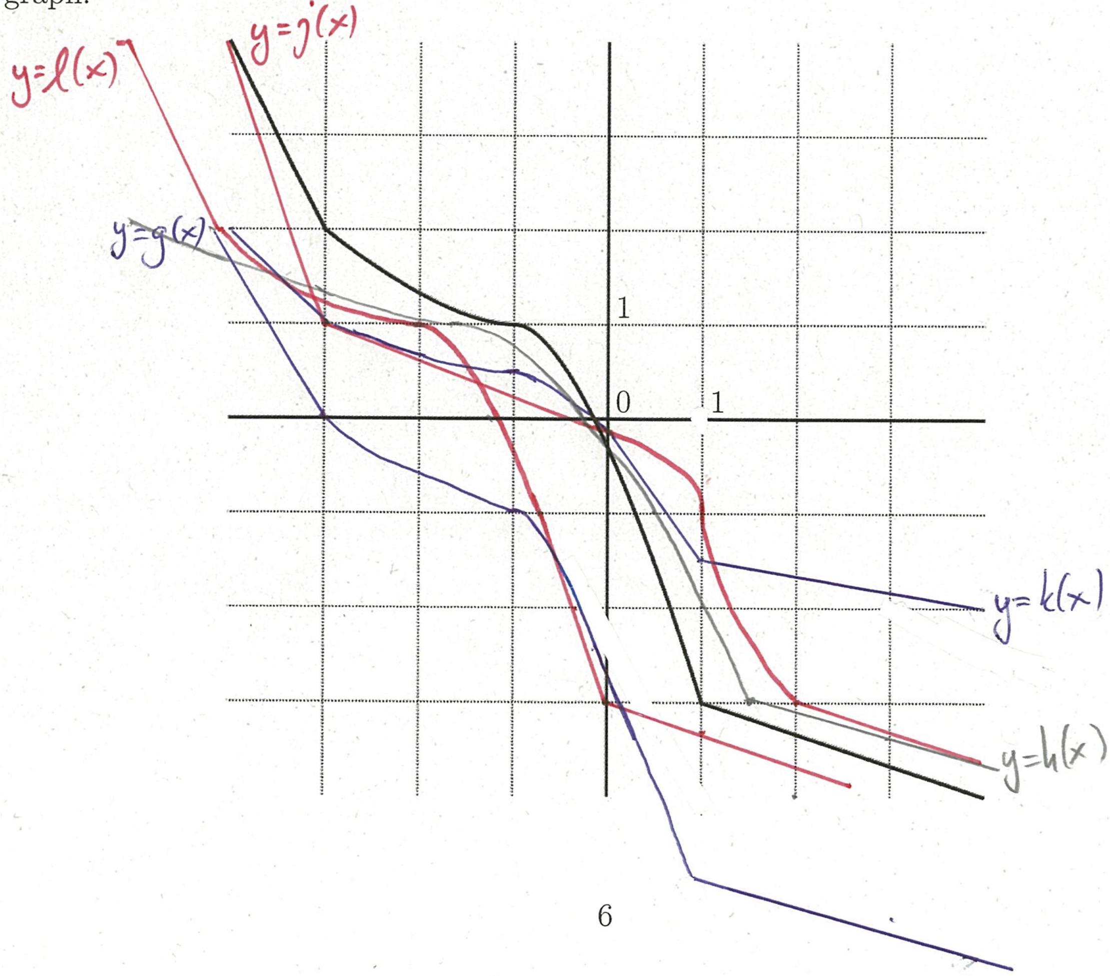
taken at $x = \frac{3}{2}$.

So the vertex is the point $\underline{\underline{\left(\frac{3}{2}, -\frac{1}{4}\right)}}$

8. Draw the graphs of f and g:



Problem 5: **(6)** We consider the function $f : [-4,4] \to [-4,4]$ with the following graph:

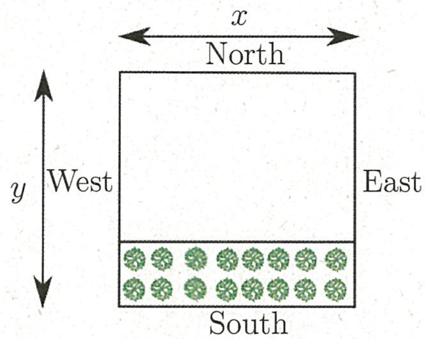


Draw the graphs of the functions defined by:

1. $g(x) = f(x) - 2$;
2. $h(x) = f(2x/3)$;
3. $j(x) = f^{-1}(x)$;
4. $k(x) = f(x)/2$;
5. $\ell(x) = f(x+1)$.

(don't forget to indicate their names clearly).

Problem 6 : (7) Mr. Tanaka wants to enclose a (rectangular) patch of land to build his house with fence. He also wants to separate a garden as indicated bellow. He bought enough wood to make 120 meters of fence. We use the following notation:



1. Compute the total length of fence he uses to make the four sides and the separation between the garden and the house as a function of x and y (there are five edges in total). Total length of fence used:
   $\ell(x,y) = 3x + 2y$

2. Express the area of the patch of land as a function of x and y.
   Area enclosed: $A(x,y) = xy$

We suppose now that Mr. Tanaka will use all the fence he got. That is, $\ell(x,y) = 120$

3. Express y as a function of x.

$$\left.\begin{array}{l}\ell(x,y) = 3x+2y \\ \ell(x,y) = 120\end{array}\right\} \quad \begin{array}{l}3x+2y = 120 \\ 2y = 120 - 3x \\ y = 60 - \frac{3}{2}x\end{array}$$

4. Express the area of the patch of land as a function of x.

$$A = A\left(x, 60-\tfrac{3}{2}x\right) = x\cdot\left(60-\tfrac{3}{2}x\right) = -\tfrac{3}{2}x^2+60x$$

5. For which x will Mr. Tanaka get the biggest possible area? What is this area?

$$A = -\tfrac{3}{2}x^2+60x = -\tfrac{3}{2}\left(x^2-40x\right) = -\tfrac{3}{2}\left((x-20)^2-(20)^2\right)$$

$$= -\tfrac{3}{2}(x-20)^2+\tfrac{3}{2}\cdot 400 = -\tfrac{3}{2}(x-20)^2+600$$

A attains its maximal value for $\underline{\underline{x=20}}$.

The maximal value of A is $\underline{\underline{600}}$

The width of the plot should be $\underline{20\,m}$,
in order to get the maximal area, $\underline{600\ m^2}$