# Contents

# §.1. Nilpotent orbits.

- $\mathfrak{g}$ : simple Lie alg. $/\mathbb{C}$
- $G$ : Lie group $/\mathbb{C}$, s.t. $\mathrm{Lie}(G) \simeq \mathfrak{g}$
- $G_{ad} := \mathrm{Img}(\mathrm{Ad}) \subset GL(\mathfrak{g})$

  $G_{ad} \cong G^0$ $\quad$ ($G$ : conn. $\Rightarrow G_{ad} \cong G$)
- $G_{sc}$ : simple conn. & conn. Lie group s.t. $\mathrm{Lie}(G) = \mathfrak{g}$.

  $\pi_1(G) = \{e\}$, $\pi_0(G) = \{*\}$
- $G$ : simple conn. & conn. $\Rightarrow G_{sc} \cong G_{ad} \simeq G$
- $G$ : conn. $\Rightarrow G \cong G_{sc}/{}^{\exists}C$ $\qquad C \subset Z(G_{sc})$

For $x \in \mathfrak{g}$, $\quad \mathcal{O}_x := \{\mathrm{Ad}(g)x \mid g \in G_{ad}\}$

Then $\quad \mathcal{O}_x \cong G_{ad}/G_{ad}^x$, Here $G_{ad}^x = \{g \in G_{ad} \mid \mathrm{Ad}(g)x = x\}$

$$\rightsquigarrow \dim \mathcal{O}_x = \dim G_{ad} - \dim G_{ad}^x$$
$$= \dim \mathfrak{g} - \dim \mathfrak{z}_{\mathfrak{g}}(x)$$

$$\boxed{\dim \mathfrak{z}_{\mathfrak{g}}(x) \geq \ell}$$ $(\ell := \mathrm{rk}(\mathfrak{g}))$

$\because$ $x \in \mathfrak{h} \subset \mathfrak{g}$ , $\mathfrak{z}_{\mathfrak{g}}(x) \supset \mathfrak{h}$

Cartan subalg.

$\rightsquigarrow$ $\dim \mathfrak{z}_{\mathfrak{g}}(x) \geq \underset{\ell}{\underset{\shortparallel}{\dim \mathfrak{h}}}$ //

**Def.** $\ell := \mathrm{rank}(\mathfrak{g})$ , $x \in \mathfrak{g}$

(1) $x$ is regular $:\Leftrightarrow$ $\dim \mathfrak{z}_{\mathfrak{g}}(x) = \ell$

(2) $x$ is subregular $:\Leftrightarrow$ $\dim \mathfrak{z}_{\mathfrak{g}}(x) = \ell + 2$

$\mathcal{N} = \mathcal{N}(\mathfrak{g}) := \{\text{nilpotent elements of } \mathfrak{g}\}$

$$G \overset{\mathrm{Ad}}{\curvearrowright} \mathcal{N}$$

**Thm 1.1**

$\mathcal{N}/G := \{G\text{-orbits of } \mathcal{N}\}$

(1) • $\# \mathcal{N}/G < \infty$

• $\forall \mathcal{O} \in \mathcal{N}/G$ $\dim \mathcal{O} \leq \dim \mathfrak{g} - \ell \in 2\mathbb{Z}_{\geq 0}$

(2) $\mathcal{O}_{\mathrm{reg}} := \{x \in \mathcal{N} \mid x : \text{regular}\} \in \mathcal{N}/G$

• $\dim \mathcal{O}_{\mathrm{reg}} = \dim \mathfrak{g} - \ell$

• $\forall \mathcal{O} \in \mathcal{N}/G$ , $\mathcal{O} \neq \mathcal{O}_{\mathrm{reg}} \Rightarrow \dim \mathcal{O} < \dim \mathfrak{g} - \ell$ .

(3) $\mathcal{O}_{sreg} := \{x \in \mathcal{N}(\mathfrak{g}) \mid x : \text{subregular}\} \in \mathcal{N}/G$

- $\dim \mathcal{O}_{sreg} = \dim \mathfrak{g} - (\ell+2)$

codim $> 2$

codim $= 2$

- $\mathcal{N} = \mathcal{O}_{reg} \sqcup \mathcal{O}_{sreg} \sqcup \bigsqcup_{\mathcal{O} \in \mathcal{N}/G} \mathcal{O}$

codim( ) $=$ $\dim \mathcal{N} - \dim(\ )$     codim$=0$

$\dim \mathcal{O} < \dim \mathfrak{g} - (\ell+2)$

## Example 1.2

- $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \qquad (n \geq 2)$
- $G_{sc} = SL_n(\mathbb{C})$
- $Z = \left\{ \begin{pmatrix} \zeta & & 0 \\ & \ddots & \\ 0 & & \zeta \end{pmatrix} \middle| \zeta^n = 1 \right\} \cong \mathbb{Z}/n\mathbb{Z}$

$\rightsquigarrow \quad G_{ad} = PSL_n(\mathbb{C}) = SL_n(\mathbb{C})/Z$

$\mathcal{N}(\mathfrak{g}) \ni \forall x$ $\qquad$ $\mathrm{size}(J_i) = n_i$

$$\mathrm{Jordan}(x) = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \qquad n_1 \geq \cdots \geq n_k$$

$$J_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad A = (a_{ij})_{i,j} \in \mathfrak{sl}_n(\mathbb{C})$$

$$(AJ)_{ij} = \sum_{k=1}^{n} a_{ik}\,\delta_{k,j+1} = a_{i,j+1}$$

$$(JA)_{ij} = \sum_{k=1}^{n} \delta_{i,k+1}\,a_{kj} = a_{i-1,j}$$

$$AJ = \begin{bmatrix} 0 & a_{11} & \cdots & a_{1(n-2)} & a_{1(n-1)} \\ 0 & a_{21} & \cdots & a_{2(n-2)} & a_{2(n-1)} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & a_{(n-1)1} & & \ddots & \vdots \\ 0 & a_{n1} & \cdots & \cdots & a_{n(n-1)} \end{bmatrix}$$

$$JA = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$a_1 := a_{11} = a_{22} = \cdots = a_{nn} = 0 \quad \because \; Tr(A) = 0$$

$$a_2 := a_{12} = a_{23} = \cdots = a_{(n-1)n}$$

$$\vdots$$

$$a_k := a_{1k} = a_{2(k+1)} = \cdots = a_{(n-k+1)n}$$

$$\vdots$$

$$a_n := a_{1n}$$

$$\rightsquigarrow \quad A = \begin{bmatrix} 0 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ & \ddots & a_2 & a_3 & & a_{n-1} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & a_3 \\ & 0 & & & \ddots & a_2 \\ & & & & & 0 \end{bmatrix}$$

$$\rightsquigarrow \quad \dim \mathcal{O}_x = \dim \mathfrak{g} - \dim \mathfrak{z}_{\mathfrak{g}}(x)$$

$$= (n^2-1) - (n-1)$$
$$= n^2 - n \quad //$$

**Prop 1.3**

(1) $\left\{\begin{matrix}\text{nilpotent orbits} \\ \text{of } \mathfrak{sl}_n(\mathbb{C})\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \{\text{partition of } n\}$

$$\mathcal{O}_x \longmapsto \mathrm{Jordan}(x)$$

(2) $\dim \mathcal{O}_x = n^2 - \sum_{i=1}^{k} (2i-1) n_i$

Here

$$\mathrm{Jordan}(x) = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}, \quad \underset{(1 \leq i \leq k)}{\mathrm{size}(J_i) = n_i}$$

$$(n_1 \geq \cdots \geq n_k)$$

**Proof of (2)**

$$\forall x \in \mathcal{N}(\mathfrak{sl}_n(\mathbb{C})) \subset \mathfrak{sl}_n(\mathbb{C}),$$

$$\mathrm{Jordan}(x) = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} =: J$$

$$\underset{(1 \leq i \leq k)}{\mathrm{size}(J_i) = n_i}, \quad n_1 \geq \cdots \geq n_k$$

$$A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{C})$$

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix}, \quad A_{ij} \in \mathrm{Mat}_{n_i \times n_j}(\mathbb{C})$$

$$AJ = JA$$

$$\rightsquigarrow \quad A_{ij} = n_i \underbrace{\left[\begin{array}{cccc|c} a_1 & a_2 & \cdots & a_{n_i} & \\ 0 & \ddots & \ddots & \vdots & \\ \vdots & 0 & \ddots & a_2 & 0 \\ 0 & \cdots & 0 & a_1 & \end{array}\right]}_{n_j}$$

$$\rightsquigarrow \dim \{ A_{ij} \in \mathrm{Mat}_{n_i \times n_j}(\mathbb{C}) \mid A_{ij} J_j = J_i A_{ij} \}$$
$$= \min\{n_i, n_j\}$$

$$\dim \mathfrak{z}(x) = \sum_{i,j} \min\{n_i, n_j\} - 1$$
$$= (n_1 + 3n_2 + \cdots + (2k-1) n_k) - 1$$
$$= \sum_{i=1}^{k} (2i-1) n_i - 1$$

$$\therefore \quad \dim \mathcal{O}_x = \dim \mathfrak{g} - \dim \mathcal{O}_x$$
$$= (n^2-1) - \left(\sum_{i=1}^{k} (2i-1)n_i - 1\right)$$
$$= n^2 - \sum_{i=1}^{k} (2i-1)k \quad /\!\!/$$

## §2. Nilpotent variety and Kleinian singularity

**Def. 2.1**

- $x \in \mathcal{O}_{sreg} := \{x \in \mathcal{N} \mid x : \text{subregular}\}$

  $\Leftrightarrow \dim \mathfrak{z}(x) = \ell + 2$

- $S$ : non-singular subvar. of $\mathfrak{g}$,

  $\dim = \ell + 2$

$S$ is transversal slice of $G.x$

$$\Leftrightarrow \quad T_x S + T_x(G.x) = T_x \mathfrak{g}$$
$$\qquad\qquad\quad \underset{\mathfrak{g}/\mathfrak{z}(x)}{\cong} \qquad\quad \underset{\mathfrak{g}}{\cong}$$

$$\Leftrightarrow \quad T_x S \cap T_x(G.x) = \{x\}$$

$\underline{\text{Th'm 2.2}}$ $\mathfrak{g}$ : Lie alg. of type A, D, E

$\mathcal{S}$ : transversal slice of $x \in \mathcal{O}_{\mathrm{sreg}}$.

then the surface $\mathcal{S} \cap \mathcal{N}$ has Kleinian singularity of type $\mathfrak{g}$.

※ Shown by Brieskorn in 1970.

$\mathbb{C}[\mathfrak{g}]$ := the alg. of polynomial function on $\mathfrak{g}$

- $G \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}$

$\curvearrowright \mathbb{C}[\mathfrak{g}]$ by $(g \cdot f)(x) = f(\mathrm{Ad}(g^{-1})x)$

$(g \in G,\ x \in \mathfrak{g})$

$$\mathbb{C}[\mathfrak{g}]^G := \{ f \in \mathbb{C}[\mathfrak{g}] \mid g \cdot f = f,\ \forall g \in G \}$$

- $\mathfrak{h} \subset \mathfrak{g}$ : Cartan. sub alg. of $\mathfrak{g}$
- $W$ : Weyl gp. assoc. to $\mathfrak{h}$

$$\mathbb{C}[\mathfrak{h}]^W = \{ f \in \mathbb{C}[\mathfrak{h}] \mid w \cdot f = f,\ \forall w \in W \}$$

$$\mathrm{res} : \mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}],\ f \mapsto f|_{\mathfrak{h}}$$

**Thm 2.3 (Chevalley)**

(1) $\mathrm{res} : \mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$

(2) $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[h_1, \dots, h_\ell]$

exponent of $\mathfrak{g}$

$\exists\, h_i$ : homog. poly.

$\deg(h_i) = m_i + 1$

**Outline of proof (1)**

(Well-definedness)

$\forall f \in \mathbb{C}[\mathfrak{g}]^G$,

$$g \cdot \mathrm{res}(f)(h) = \mathrm{res}(f)(g^{-1} h g) = \mathrm{res}(f)(h)$$
$$(g \in G,\ h \in \mathfrak{h})$$

$\rightsquigarrow\ g \in N_G(T)$ $\because$ $g^{-1} h g = h$.

$(\mathfrak{h} = \mathrm{Lie}(T))$

$w = gT \in N_G(T) \cong W$

$\leadsto \quad w.\mathrm{res}(f)(h) = \mathrm{res}(f)(w.h)$
$= \mathrm{res}(f)(h)$

$\therefore \quad \mathrm{res}(f) \in \mathbb{C}[\mathfrak{h}]^W.$

(injectivity)

$f \in \mathbb{C}[\mathfrak{g}]^G, \ \mathrm{res}(f) = 0 \Longrightarrow f = 0$

use

$\mathcal{O}_{reg}$ is dense in $\mathfrak{g}$

(surjectivity)

- $\mathfrak{g} = \mathrm{Lie}(G)$
- $\rho$ : a rep. of $G$
- $\chi_\rho$ : character of $\rho$

$\leadsto \quad \chi_\rho \in \mathbb{C}[G]^G$

- $\rho_\lambda$ : irr. rep. of h.w. $\lambda$
- $\chi_\lambda := \chi_{\rho_\lambda}$

$$x_\lambda|_T \in X(T)^W := \mathrm{Hom}(T, \mathbb{C}^*)$$

$$\rightsquigarrow \quad x_\lambda|_T = [\lambda] + \sum_{\mu<\lambda} n_\mu [\mu] \quad (n_\mu \in \mathbb{Z}_{\geq 0})$$

$$(W.\mu \cap P_+ = \{\mu_+\})$$

$$\rightsquigarrow \quad [\lambda] = x_\lambda|_T + \sum_{\mu:\text{dominant}} m_\mu x_\mu|_T \quad (m_\mu \in \mathbb{Z})$$

- $\{\omega_i \in P_+ \mid \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}\}$ ; basis of $X(T)$.

$\rightsquigarrow$ $\{x_{\omega_i}\}$ are generators of $\mathbb{C}[G]^G$,

algebraic independent over $\mathbb{C}$.

//

**Th'm 2.4.**

For $x \in \mathfrak{g}$, we take a Jordan decomp.

$$x = x_s + x_n$$

$\leftarrow$ nilpotent part

$\uparrow$ semisimple part

For $f \in \mathbb{C}[\mathfrak{g}]^G$, we have

$$f(x) = f(x_s)$$

Ex. 2.5

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $A \in \mathfrak{sl}_n(\mathbb{C})$.

- $\det(tI_n - A) = t^n + \chi_2(A)t^{n-1} + \cdots + (-1)^n \chi_n(A)$

$\chi_k \in \mathbb{C}[\mathfrak{g}]^G \quad (1 \leq k \leq n)$

$\because$ $\forall X \in G = SL_n(\mathbb{C})$,

$\det(tI_n - X^{-1}AX)$

$= t^n + \chi_2(X^{-1}AX)t^{n-1} + \cdots + (-1)^n \chi_n(X^{-1}AX)$

On the other hand,

$$\begin{aligned}\det(tI_n - X^{-1}AX) &= \det(X^{-1}tI_nX - X^{-1}AX) \\ &= \det(X^{-1}(tI_n - A)X) \\ &= \det(X^{-1})\det(X)\det(tI_n - A) \\ &= \det(tI_n - A)\end{aligned}$$

$\therefore \quad \chi_k(A) = \chi_k(X^{-1}AX)$ //

$\mathfrak{S}_n \curvearrowright \mathfrak{h} = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\} \subset \mathfrak{sl}_n(\mathbb{C})$

(Cartan)

$\cup\!\!\!\!\!\;|$

$(h_1, \ldots, h_n)$ $\leftarrow$ coordinate.

$e_k$ : fundamental symm. poly. ($\deg = k$)

$$e_1 = h_1 + \cdots + h_n = 0$$

$$\rightsquigarrow \quad \mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[e_2, \cdots, e_n]$$

$\chi_k(A)$ is fundamental symm. poly $\deg = k$ of eigenvalue $\lambda_1, \ldots, \lambda_n$ of $A$.

$$\mathrm{res} : \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W$$

$$\chi_k(\mathrm{Jordan}(A)) \longmapsto e_k$$

**Thm. 2.6.**

(1) semisimple element $x$ is conjugate w/ $\mathfrak{h}$

(2) $x, y \in \mathfrak{h}$ are $G$-conj.

$$\Longrightarrow \quad W \cdot x = W \cdot y$$

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\chi_1, \ldots, \chi_l]$$

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{h}/W \quad (\cong \mathbb{C}^l)$$

$$x \longmapsto (\chi_1(x), \ldots, \chi_l(x))$$

adjoint quotient.

Th'm 2.?.

(1) $\chi$ is flat morphism, codimension of fiber $= l$

(2) $\chi^{-1}(\bar{h}) = \bigsqcup_{i=0}^{k} G.x_i \quad (\bar{h} \in \mathfrak{h}/W)$

(3) $x_0$ : semi simple, $\operatorname{codim}_{\chi^{-1}(\bar{h})} (G.x_i) \geq 2 \quad (i \neq 0)$

(4) $\{x \in \chi^{-1}(\bar{h}) \mid x : \text{non sing. pt.}\}$

$= \{x \in \chi^{-1}(\bar{h}) \mid x : \text{regular (as } \mathfrak{g})\}$

$X, Y$ : algebraic var.

$X, Y$ : non-singular

$\Rightarrow$ $f: X \to Y$ is flat

$:\Longleftrightarrow$ $\dim f^{-1}(y)$ : constant $({}^{\forall}y \in Y)$

$A, B$ : comm. ring $\left[\begin{array}{l} X = \operatorname{Spec} A \\ Y = \operatorname{Spec} B \end{array}\right]$

$\varphi: B \to A$ : ring hom. $[f: X \to Y]$
morphism

$\varphi$ : flat

$:\Longleftrightarrow$ $\forall$ $0 \to L \to M \to N \to 0$ (exact in Mod B)

$0 \to L \otimes_B A \to M \otimes_B A \to N \otimes_B A \to 0$ (exact in Mod A)

## §3. Representation of $\mathfrak{sl}_2(\mathbb{C})$ and transversal slice.

$\mathfrak{g}$ : simple Lie alg.

$\cup$

$\mathcal{O}_{sreg} = \{ x \in \mathcal{N} \mid x : \text{subregular} \}$

$\dim \mathfrak{z}_{\mathfrak{g}}(x) = \operatorname{rank}(\mathfrak{g}) + 2$

$= \mathrm{Ad}(G)(x) \quad (x \in \mathcal{O}_{sreg})$

$\rightsquigarrow T_x(\mathcal{O}_{sreg}) = x + [\mathfrak{g}, x]$

$\mathcal{S} = x + V$ : transversal slice

$T_x \mathcal{S} + T_x \mathcal{O}_{sreg} = T_x \mathfrak{g}$

$\rightsquigarrow \quad T_x \mathfrak{g} = V \oplus [\mathfrak{g}, x]$ (as vector sp.)

construct.

**Th'm 3.1** (Jacobson–Morozov)

$\mathfrak{g}$ : semi-simple.

For $x \in \mathcal{N} \subset \mathfrak{g}$, there exists $\exists!$ $h, y \in \mathfrak{g}$ that satisfy the following rel.

$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h$

up to adjoint action.

Thm 3.2.

$$\{x \in \mathcal{N} \mid x \neq 0\}/G \overset{1:1}{\longleftrightarrow} \{\mathfrak{a} \subset \mathfrak{g} \mid \mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C})\}$$

$$\mathcal{O}_x \longmapsto \mathfrak{a} = \underline{\langle x, h, y \rangle} \quad (x \in \mathfrak{a})$$

$sl_2$-triple

Well-definedness

$$\mathcal{O}_x = \mathcal{O}_{x'} \quad (x' = g x g^{-1})$$

$$\rightsquigarrow \mathfrak{a} = \langle x, h, y \rangle, \ \mathfrak{a}' = \langle x', h', y' \rangle$$

- $h' = g h g^{-1}$
- $y' = g y g^{-1}$

//

- $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathfrak{sl}_2(\mathbb{C})$
- $\rho : \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathrm{End}(V)$ : representation of $\mathfrak{sl}_2(\mathbb{C})$

- $V_\lambda := \{v \in V \mid \rho(h) v = \lambda v\}$

※ $V_\lambda \neq \{0\} \rightsquigarrow$
- $\lambda$ : a weight of $h$
- $V_\lambda$ : a weight space.

$V = \bigoplus_\lambda V_\lambda$ : weight space decomposition

<u>Th'm 3.3.</u>

(1) $\mathbb{Z}_{>0} \overset{1:1}{\longleftrightarrow} \{\text{irr. rep. of f.d of } \mathfrak{sl}_2(\mathbb{C})\}/\cong$

$\quad\; n \longmapsto V^n$ : $\dim(V_n) = n$.

(2) • $\text{weight}(V^n) = \{-(n-1), -(n-3), \cdots, n-1\}$

↑ set of weight of $V_n$

• $\dim(V^n_\lambda) = 1 \quad {}^\forall \lambda \in \text{weight}(V^n)$

• $\rho(x) : V^n_\lambda \xrightarrow{\sim} V^n_{\lambda+2} \qquad \lambda \leq n-3$

$\rho(y) : V^n_\lambda \xrightarrow{\sim} V^n_{\lambda-2} \qquad \lambda \geq -(n-3)$

※ $\lambda = \begin{cases} n-1 & : \text{highest weight.} \\ -(n-1) & : \text{lowest weight.} \end{cases}$

$\rho : sl_2(\mathbb{C}) \longrightarrow \mathrm{End}(\mathbb{C}[z_1, z_2])$

- $\rho(h) = z_1 \partial_{z_1} - z_2 \partial_{z_2}$
- $\rho(x) = z_1 \partial_{z_2}$
- $\rho(y) = z_2 \partial_{z_1}$

$R^n := \{ \text{homogenious polynomials, } \deg = n-1 \}$
$\quad = \langle z_1^{n-1}, z_1^{n-2} z_2, \cdots, z_2^{n-1} \rangle \subset \mathbb{C}[z_1, z_2]$

Then $R^n$ is irreducible rep. of $sl_2(\mathbb{C})$.
$\left( \begin{smallmatrix} \cong \\ V^n \end{smallmatrix} \right)$

$\mathfrak{a} \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{End}(\mathfrak{g})$ : hom.
$\left( \begin{smallmatrix} \cong \\ sl_2(\mathbb{C}) \end{smallmatrix} \right)$

Since any f.d. rep. of $sl_2(\mathbb{C})$ is complete reducible,
we obtain the following

$$\mathfrak{g} = \bigoplus_{i=1}^{r} V^{n_i} \quad : V^{n_i} : \text{irr. rep of } \mathfrak{a}.$$

$v_i$ : weight vector having lowest weight with respect to $h$ in $V^{n_i}$.

$\rightsquigarrow \ker(\mathrm{ad}(y))|_{V^{n_i}} = \mathbb{C}v_i$

$\left( \mathfrak{z}_{V^{n_i}}(y) = \{ v \in V^{n_i} \mid [y,v]=0 \} = \mathbb{C}v_i \right)$

By Th'm 3.3 (2), we obtain the following.

$$V^{n_i} = \mathrm{ad}(x)(V^{n_i}) \oplus \mathbb{C}v_i$$
$$= \mathrm{ad}(x)(V^{n_i}) \oplus \mathfrak{z}_{V^{n_i}}(y)$$

$$\therefore \quad \mathfrak{g} = \mathrm{ad}(x)(\mathfrak{g}) \oplus \mathfrak{z}_{\mathfrak{g}}(y)$$

- $\mathcal{S} := x + \mathfrak{z}_{\mathfrak{g}}(y) \subset \mathfrak{g}$ (Slodwy slice)

Compliment space of $[\mathfrak{g}, x]$ in $\mathfrak{g}$.

<u>Th'm 3.4.</u>

$x \in \mathcal{O}_{sreg}$, $x \in \exists\, \mathfrak{a} < \mathfrak{g}$, $\mathfrak{a} = \langle x, h, y \rangle \cong \mathfrak{sl}_2(\mathbb{C})$

$\mathcal{S} = x + \mathfrak{z}_{\mathfrak{g}}(y)$ and $\mathcal{O}_{sreg}$ intersect transversal at $x$.

## Example 3.5.

$$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$$

$$S := \left\{ \left[\begin{array}{cc|c} 0 & 1 & 0 \\ X_2 & X_1 & Y \\ \hline Z & 0 & -X_1 \end{array}\right] \;\middle|\; (X_1, X_2, Y, Z) \in \mathbb{C}^4 \right\}$$

$$\ni x = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$\rightsquigarrow$ $S$ and $\mathcal{O}_{\mathrm{sreg}}$ intersect transversal at $x \in \mathcal{O}_{\mathrm{sreg}}$.

$\because$

$$\begin{array}{ccccc} T_x S & + & T_x \mathcal{O}_{\mathrm{sreg}} & \overset{?}{=} & T_x(\mathfrak{g}) \\ \| & & \| & & \| \\ S & + & x + [\mathfrak{g}, x] & & x + \mathfrak{g} \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ (S - x) & \oplus & [\mathfrak{g}, x] & \overset{?}{=} & \mathfrak{g} \\ \| & & \| & & \| \\ \left\{\begin{pmatrix} 0 & 0 & 0 \\ * & X_1 & * \\ * & * & -X_1 \end{pmatrix}\right\} & & \left\{\begin{bmatrix} g & * & * \\ 0 & -g & 0 \\ 0 & * & 0 \end{bmatrix}\right\} & & \left\{\begin{bmatrix} a & * & * \\ * & b & * \\ * & * & -(a+b) \end{bmatrix}\right\} \end{array}$$

$\varphi$ : "$-x$"

$\forall$

$$y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \in \mathfrak{g}$$

$$yx = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & y_{11} & 0 \\ 0 & y_{21} & 0 \\ 0 & y_{31} & 0 \end{bmatrix}$$

$$xy = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

$$= \begin{bmatrix} y_{21} & y_{22} & y_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \quad xy - yx = \begin{bmatrix} y_{21} & y_{22} - y_{11} & y_{23} \\ 0 & -y_{21} & 0 \\ 0 & -y_{31} & 0 \end{bmatrix}$$

$$\rightsquigarrow \quad [x, \mathfrak{g}] = \left\{ \begin{bmatrix} g & * & * \\ 0 & -g & 0 \\ 0 & * & 0 \end{bmatrix} \right\}$$

## §4. Action on $\mathbb{C}^*$ on transversal slice

$\mathfrak{g}$ : Lie alg. type A, D, E

$\cup$

$x \in \mathcal{N}$ : nilpotent var.

$\mathfrak{a} = \langle \underline{x, h, y} \rangle \subset \mathfrak{g}$ (∵ Jacobson - Morozov).

$\uparrow$ $\mathfrak{sl}_2$-triple

$\mathcal{S} = x + \mathfrak{z}_{\mathfrak{g}}(y)$ : Slodowy slice.

Then $\mathcal{S}$ is a versal deformation of Kleinian sing.

表 3.5 半普遍変形の重み

| $\widetilde{\Gamma}$ | 型 | $w_1$ | $w_2$ | $w_3$ | $\delta_1$ | $\delta_2$ | $\cdots$ | $\delta_{l-1}$ | $\delta_l$ | $d$ | $l$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_n$ | $A_{n-1}$ | 2 | $n$ | $n$ | 4 | 6 | $\cdots$ | $2(n-1)$ | $2n$ | $2n$ | $n-1$ |
| $\widetilde{\mathcal{D}}_n$ | $D_{n+2}$ | 4 | $2n$ | $2(n+1)$ | 4 | 8 | $12, \cdots, 4n$ | $2(n+2)$ | $4(n+1)$ | $4(n+1)$ | $n+2$ |
| $\widetilde{\mathcal{T}}$ | $E_6$ | 6 | 8 | 12 | 4 | 10 | 12, 16 | 18 | 24 | 24 | 6 |
| $\widetilde{\mathcal{O}}$ | $E_7$ | 8 | 12 | 18 | 4 | 12 | 16, 20, 24 | 28 | 36 | 36 | 7 |
| $\widetilde{\mathcal{I}}$ | $E_8$ | 12 | 20 | 30 | 4 | 16 | 24,28,36, 40 | 48 | 60 | 60 | 8 |

$$A_{n-1} : F = x^n + yz + a_1 x^{n-2} + \cdots + a_{n-1}$$

$$D_{n+2} : F = x^{n+1} + xy^2 + z^2 + a_1 x^n + a_2 x^{n-1} + \cdots + a_{n+2} + 2a_{n+1} y$$

$$E_6 : F = x^4 + y^3 + z^2 + a_1 x^2 y + a_2 xy + a_3 x^2 + a_4 y + a_5 x + a_6$$

$$E_7 : F = x^3 y + y^3 + z^2 + a_1 x^4 + a_2 x^3 + a_3 xy + a_4 x^2 + a_5 y + a_6 x + a_7$$

$$E_8 : F = x^5 + y^3 + z^2 + a_1 x^3 y + a_2 x^2 y + a_3 x^3 + a_4 xy + a_5 x^2 + a_6 y + a_7 x + a_8$$

表 4.2　$\mathbf{C}^\times$ 作用の重み

| | $d_1$ | $d_2$ | $d_3$ | $\cdots$ | | | $d_{l-1}$ | $d_l$ | $n_1$ | $n_2$ | $\cdots$ | | | | $n_{l-1}$ | $n_l$ | $n_{l+1}$ | $n_{l+2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_l$ | 2 | 3 | 4 | $\cdots$ | | | $l$ | $l+1$ | 3 | 5 | $\cdots$ | | | | $2l-1$ | 1 | $l$ | $l$ |
| $D_l$ | 2 | 4 | 6 | $\cdots$ | $2l-6$ | $2l-4$ | $l$ | $2l-2$ | 3 | 7 | $\cdots$ | | | $4l-9$ | $2l-1$ | 3 | $2l-5$ | $2l-3$ |
| $E_6$ | 2 | 5 | 6 | | | 8 | 9 | 12 | 3 | 9 | 11 | | | 15 | 17 | 5 | 7 | 11 |
| $E_7$ | 2 | 6 | 8 | | 10 | 12 | 14 | 18 | 3 | 11 | 15 | | 19 | 23 | 27 | 7 | 11 | 17 |
| $E_8$ | 2 | 8 | 12 | 14 | 18 | 20 | 24 | 30 | 3 | 15 | 23 | 27 | 35 | 39 | 47 | 11 | 19 | 29 |

$\rightsquigarrow$ (1) $n_i = 2d_i - 1 \qquad i \leq l-1$

(2) • $\delta_i = 2d_i = n_i + 1 \qquad (i \leq l-1)$

• $\delta_l = 2d_l$

• $w_i = n_{l+i-1} + 1 \qquad (i = 1, 2, 3)$

$\mathfrak{g}$ : A, D, E / $\mathbb{C}$ , $\operatorname{rank}(\mathfrak{g}) = l$

For $x \in \mathcal{O}_{\mathrm{sreg}}$ , $\quad \mathfrak{s} = \langle x, h, y \rangle \subset \mathfrak{g}$

$$\left( \begin{array}{c} \cong \\ \mathfrak{sl}_2(\mathbb{C}) \end{array} \right)$$

$$\mathcal{S} = x + \mathfrak{z}_{\mathfrak{g}}(y) \quad , \quad \dim \mathfrak{z}_{\mathfrak{g}}(y) = l + 2$$

$$\rightsquigarrow \quad y \in \mathcal{O}_{\mathrm{sreg}} .$$

$V^{n_i}$ : irreducible rep. of $\mathfrak{s}$, $\dim = n_i$

$v_i$ : lowest weight vector of $V^{n_i}$

$$\operatorname{ad}(h) \big( \underbrace{x + \sum_{i=1}^{l+2} c_i v_i}_{\in \mathcal{S}} \big) = 2x + \sum_{i=1}^{l+2} (-n_i + 1) c_i v_i$$

$$\begin{array}{rccc} \lambda : & \mathbb{C}^* & \longrightarrow & GL(\mathfrak{g}) \\ & \rotatebox{90}{$\in$} & & \\ & t & \longmapsto & \lambda(t)(v) = t^k v \end{array} \qquad (\mathrm{wt}(v) = k)$$

$$\left( \mathfrak{g} = \bigoplus_{i=1}^{r} V^{n_i} \right)$$

For $\forall\, x + \sum_{i=1}^{l+2} c_i v_i \in \mathcal{S}$

$$\lambda(t)\left(x+\sum_{i=1}^{\ell+2} c_i v_i\right)$$

$$= t^2 x + \sum_{i=1}^{\ell+2} t^{-n_i+1} c_i v_i$$

<u>Recall</u>

$$\mathrm{wt}(V^{n_i}) = \{\underbrace{-(n_i-1)}_{\ell.w.}, -(n_i-3), \cdots, n_i-3, \underbrace{n_i-1}_{h.w}\}$$

$$\sigma : \mathbb{C}^\times \longrightarrow GL(\mathfrak{g})$$

$$t \longmapsto \sigma(t)(v) = tv \qquad (v \in \mathfrak{g})$$

$\rightsquigarrow$ $\lambda$ and $\sigma$ are commutative.

- $\mu(t) := \sigma(t^2)\lambda(t^{-1}) \qquad (t \in \mathbb{C}^\times)$

$\rightsquigarrow$ $\mathbb{C}^\times \overset{\mu}{\curvearrowright} S$

$\because$ $\mu(t)\left(x+\sum_{i=1}^{\ell+2} c_i v_i\right) = \sigma(t^2)\left(t^{-2}x+\sum_{i=1}^{\ell+2} t^{n_i-1} c_i v_i\right)$

$$= x + \sum_{i=1}^{\ell+2} t^{n_i+1} c_i v_i \in S$$

//

## Recall

$$\chi : \mathfrak{g} \to \mathfrak{h}/W \ (\cong \mathbb{C}^{\ell}) \qquad \text{(adjoint quotient)}$$

$$x \mapsto (\chi_1(x), \dots, \chi_{\ell+2}(x))$$

where $\chi_1, \dots, \chi_{\ell+2}$ are generators of $\mathbb{C}[\mathfrak{g}]^G$.

$$\deg \chi_i = m_i + 1 =: d_i$$

exponent of $\mathfrak{g}$.

c.f. $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\cup$

$\mathfrak{h} = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n] \ (\cong \mathbb{A}^n_{\mathbb{C}})$

$\mathfrak{h}/W := \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]^W \quad (W \cong \mathfrak{S}_n)$

$\mathbb{C}[x_1, \dots, x_n]^W \underset{\text{chevalley}}{\simeq} \mathbb{C}[\chi_1, \dots, \chi_n]$

$\chi_i = e_i$ (fundamental symm. poly, $\deg = i$)

Prop 4.1

$$x_S := x|_S : S \to \mathfrak{h}/W$$

is $\mathbb{C}^*$-map with weight
$(2d_1, \dots, 2d_\ell \,;\, n_1+1, \dots, n_{\ell+2}+1)$

$(v_1, \dots, v_{\ell+2})$ : coordinate of $S$
$(x_1, \dots, x_\ell)$ : coordinate of $\mathfrak{h}/W \left(\simeq \mathbb{A}^\ell_{\mathbb{C}}\right)$
$\forall t \in \mathbb{C}^*$

$$\begin{tikzcd}
v_i \arrow[d, mapsto] & S \arrow[r, "x"] \arrow[d, "\mu(t)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{h}/W := \operatorname{Spec} \mathbb{C}[x_1, \dots, x_\ell] \arrow[d] & x_i \arrow[d, mapsto] \\
t^{n_i+1} v_i & S \arrow[r, "x"] & \mathfrak{h}/W & t^{2d_i} x_i
\end{tikzcd}$$

$\because$ ${}^\forall z \in S$

$$\begin{aligned}
x_i(\mu(t) z) &= x_i(\sigma(t^2) \lambda(t^{-1}) z) \\
&= x_i(\sigma(t^2) z) \quad \because x \text{ is inv. on Ad.} \\
&= t^{2d_i} x_i(z) \quad \because \deg(x_i) = d_i
\end{aligned}$$

Cf.

$$\lambda(t)(v) = t^k v \qquad (v \in V^k)$$

$$\left.\frac{d(\lambda(t)(v))}{dt}\right|_{t=1} = \left.\frac{d(t^k v)}{dt}\right|_{t=1} = k t^{k-1} v\Big|_{t=1}$$

$$= kv = \mathrm{ad}(h)(v)$$

$$\therefore \quad \left.\frac{d\lambda(t)}{dt}\right|_{t=1} = \mathrm{ad}(h) = \mathrm{ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\Big\uparrow \left.\frac{d}{dt}\right|_{t=1} \qquad\qquad \Big\uparrow \left.\frac{d}{dt}\right|_{t=1}$$

$$\lambda(t) = \mathrm{Ad}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)$$

$$\mathbb{C} \simeq \mathbb{C}.h \xrightarrow[\exp]{} \begin{bmatrix} e^c & 0 \\ 0 & e^{-c} \end{bmatrix} \qquad t = e^c \in \mathbb{C}^* \quad (c \in \mathbb{C})$$

<u>Example</u>

$A_n$ : $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$

$\{x, h, y\}$ : $\mathfrak{sl}_2$-triple s.t.

$$h \in C := \{h' \in \mathfrak{h} \mid \alpha_i(h') \geq 0, \forall i\}$$

$\Pi = \{\alpha_1, \cdots, \alpha_n\}$ ; simple root.

$n = 2k+1$

$\{d_i\} = \{2, \dots, n+1\} = \{2, \dots, 2k+2\}$

<u>Fact</u> $\quad \alpha_i(h) = \{0, 1, 2\}$

$$x = \left[\begin{array}{cccc|c} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & 0 \\ & & & 0 & \\ \hline & 0 & & & 0 \end{array}\right], \quad h = \left[\begin{array}{cccc|c} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & 0 \\ & & & -(n-1) & \\ \hline & 0 & & & 0 \end{array}\right]$$

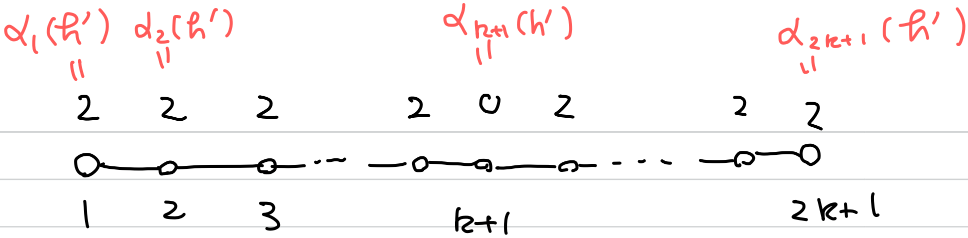
$$y = \left[\begin{array}{cccc|c} 0 & & & & \\ y_1 & \ddots & & & \\ & \ddots & \ddots & & 0 \\ & & y_{n-1} & 0 & \\ \hline & 0 & & & 0 \end{array}\right] \qquad y_i = i(n-i)$$

$\rightsquigarrow h \notin C$

$$\begin{array}{ccccccccccccc} & 1 & 2 & 3 & & k & k+1 & k+2 & k+3 & & 2k+2 & \\ h' = \mathrm{diag}( & 2k, & 2k-2, & 2k-4, & \dots, & 2, & 0 & 0, & -2, & \dots, & -2k & ) \end{array}$$

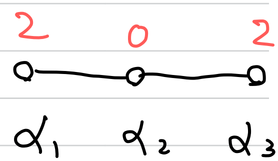
$= (k+2, k+3, \dots, 2k+2) \cdot h$

cyc. elem. in $W = \mathfrak{S}_{n+1} = \mathfrak{S}_{2k+2}$




$\mathfrak{g} = sl_4(\mathbb{C})$

$$\begin{bmatrix} 0 & \overset{2}{\alpha_1} & \overset{2}{\alpha_1+\alpha_2} & \overset{4}{\alpha_1+\alpha_2+\alpha_3} \\ & & \underset{0}{\alpha_2} & \underset{2}{\alpha_2+\alpha_3} \\ & & & \overset{2}{\alpha_3} \\ & & & 0 \end{bmatrix}$$




| Wt. | # |
|---|---|
| 4 | 1 |
| 2 | 4 |
| 0 | 1+3+1 |
| −2 | 4 |
| −4 | 1 |

$\exists\, V^5 \longrightarrow \boxed{\begin{matrix} 4 \\ 2 \\ 0 \\ -2 \\ -4 \end{matrix}}$

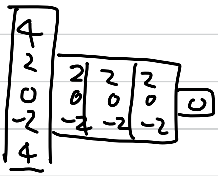
$\rightsquigarrow \#\{2\} = 4-1 = 3$

$\exists\ V_1^3,\ V_2^3,\ V_3^3$

$\rightsquigarrow \boxed{\begin{matrix} 2 \\ 0 \\ -2 \end{matrix}}\ \boxed{\begin{matrix} 2 \\ 0 \\ -2 \end{matrix}}\ \boxed{\begin{matrix} 2 \\ 0 \\ -2 \end{matrix}}$

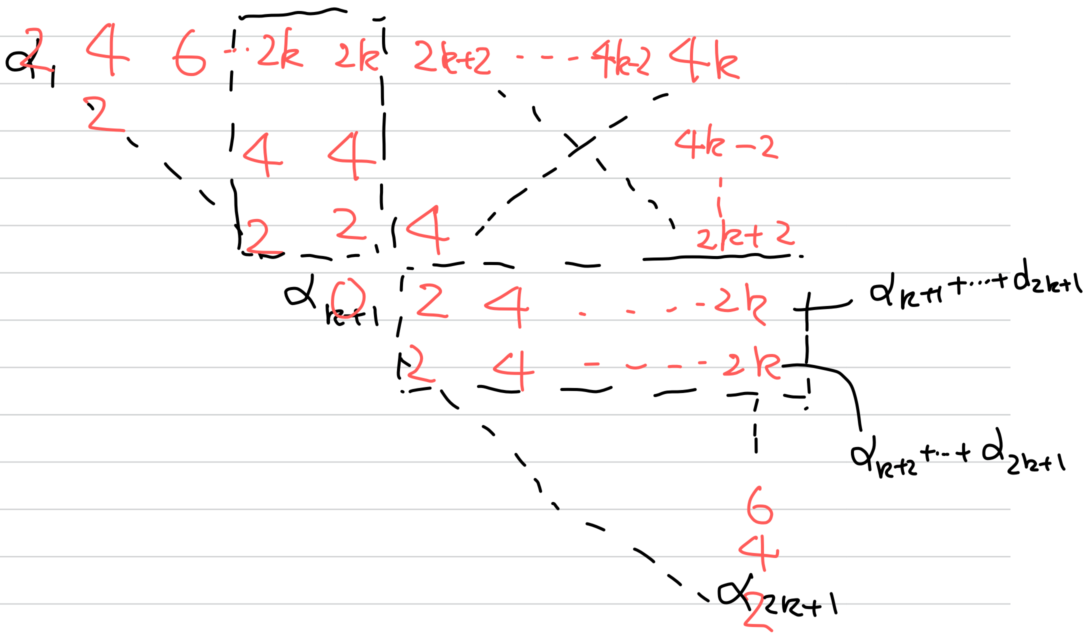
$\rightsquigarrow \quad \#\{0\} = 5 - 1 - 3 = 1$

$\exists\, V^1 \quad \boxed{0}$

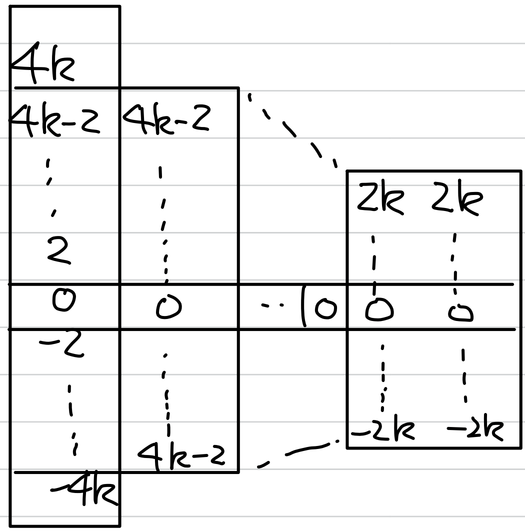



$(n_1, n_2, n_3, n_4, n_5) = (3, 5, 1, 3, 3)$

$\mathfrak{sl}_4(\mathbb{C}) = V^5 \oplus V_1^3 \oplus V_2^3 \oplus V_3^3 \oplus V^1$



| Wt | # |
|---|---|
| $4k$ | $1$ |
| $4k-2$ | $2$ |
| $4k-4$ | $3$ |
| $\vdots$ | $\vdots$ |
| $2k+2$ | $k$ |
| $2k$ | $k+3 = k+1+2$ |
| $2k-2$ | $k+2+2$ |
| $\vdots$ | |
| $0$ | $2k+1+2$ |




$$\underbrace{V^{4k+1}\ V^{4k-1}\ \cdots\ V^{1}}_{\substack{2k+1 \\ \parallel \\ n}}\ \underbrace{V^{2k-1}\ V^{2k-1}}_{2}$$

$$n_{\ell-1}\ n_{\ell-2} \cdots n_1\ n_\ell\ n_{\ell+1}\ n_{\ell+2}$$

## §5. Transversal slice and Kleinian singularity

$\mathfrak{g}$ : Lie alg. type A, D, E

**Prop. 5.1**

For each $z \in \mathcal{S}$,

$\mathcal{S}$ and $\mathcal{O}_z$ intersect transversality at $z$.

**Proof.**

$$\nu : G \times \mathcal{S} \longrightarrow \mathfrak{g}$$
$$(g, s) \longmapsto g.s = \mathrm{Ad}(g)s$$

$\mathbb{C}^* \curvearrowright G \times \mathcal{S}$

by $\rho(t)(g, s) = (\lambda(t^{-1}) g \lambda(t), \mu(t) s)$

$(t \in \mathbb{C}^*)$

$\lambda(t) \in G_{ad} = \{ \mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g} \mid g \in G \}$

$$\begin{tikzcd} G \times \mathcal{S} \arrow[r, "\nu"] \arrow[d, "\rho(t)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[d, "\mu(t)"] \\ G \times \mathcal{S} \arrow[r, "\nu"'] & \mathfrak{g} \end{tikzcd}$$

$\nu(\rho(t).(g,s))$

$= \nu((\lambda(t^{-1})g\lambda(t), \mu(t)s))$

$= \nu((\lambda(t^{-1})g\lambda(t), \sigma(t^2)\lambda(t^{-1})s))$

$= \mathrm{Ad}(\lambda(t^{-1})g\lambda(t))(\sigma(t^2)\lambda(t^{-1})s)$

$= \mathrm{Ad}(\lambda(t^{-1})g\lambda(t))(\lambda(t^{-1})\sigma(t^2)s)$

$= \mathrm{Ad}(\lambda(t^{-1})\sigma(t^2)g)(s)$

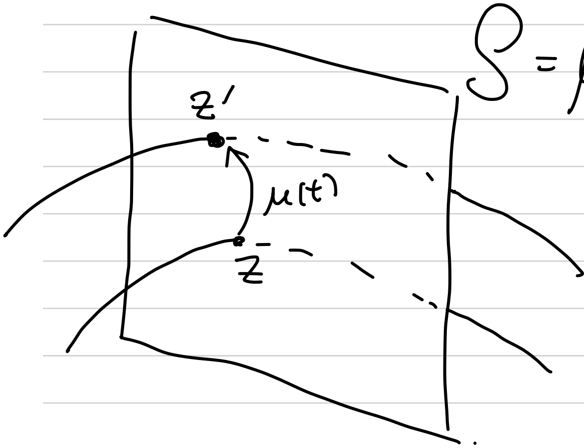
$= \mu(t)g.s$

Suppose $\mathcal{O}_z$ and $\mathcal{S}$ intersect transvarsality at $z \in \mathcal{S}$

$\mathcal{O}_z = \nu(G \times \{z\})$

i.e.

$$T_z \mathcal{O}_z \cap T_z S = \{0\} \quad \cdots (*)$$




$$S = \mu(t) S$$

$$\mathcal{O}_{z'} = \mu(t) \mathcal{O}_z$$

$$\mathcal{O}_z$$

∵ $\mu(t)\mathcal{O}_z$
$= \mu(t)\left(\nu(G \times \{z\})\right)$
$= \nu\left(\rho(t)(G \times \{z\})\right)$
$= \nu\left(\lambda(t)^{-1} G \lambda(t), \{\mu(t)z\}\right)$
$= \mathcal{O}_{z'}$

$$(*) \implies T_{z'} \mathcal{O}_{z'} \cap T_{z'} S = \{0\}$$

∵ $T_{z'} \mathcal{O}_{z'} \cap T_{z'} S$

$= T_{\mu(t)z} \mathcal{O}_{\mu(t)z} \cap T_{\mu(t)z} S$

$= \mu(t) \left( T_z \mathcal{O}_z \cap T_z S \right)$

$= \mu(t)(\{0\}) = \{0\}$

$t \in \mathbb{C}^* = GL(1) \overset{\mu}{\curvearrowright} Y$ sm.

$$\overset{\mu}{\curvearrowright} T_y Y$$

For $(g,s) \in G \times \mathcal{S}$,

$$\exists\, (g,s) \in U \underset{op}{\subset} G \times \mathcal{S}$$

$$\nu(U) = \{\nu(g',s') \mid (g',s') \in U\}$$

$$= \{Ad(g')(s') \mid (g',s') \in U\}$$

$$\begin{array}{ccccc} \dim & \dim G + \dim \mathcal{S} & & \dim \mathcal{S} & \dim G = \dim \mathfrak{g} \\ & U & \cong & \mathbb{C}^m & \times\ \nu(U) \end{array}$$

$$m = \dim(G \times \mathcal{S}) - \dim \mathfrak{g}$$

$$= \cancel{\dim G} + \dim \mathcal{S} - \cancel{\dim \mathfrak{g}}$$

$$= \dim \mathcal{S}$$

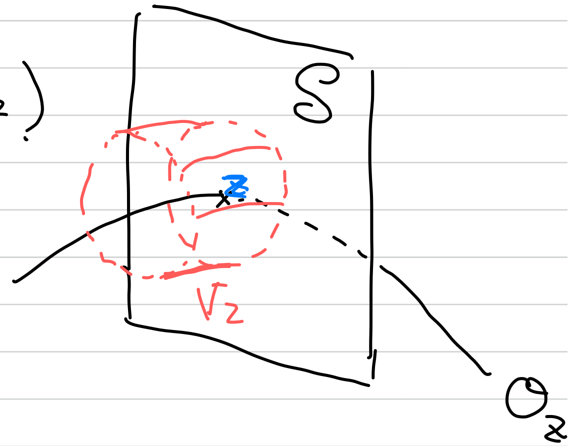
$z = \nu(e,z)$, $e \in V_1 \underset{op}{\subset} G$, $V_2 \underset{op}{\subset} \mathcal{S}$ s.t. $U = V_1 \times V_2$

$$\rightsquigarrow T_z \mathcal{O}_z \cap T_z \mathcal{S} = \{0\}$$

$$\Longrightarrow \dim U - \dim V_2 = \dim \nu(U)$$

$$\Longleftrightarrow U \cong \nu(U) \times V_2$$

$$\begin{array}{ccc} \{e\}\times V_z & \cong & \nu(\{e\}\times V_z) \\ \cap & & \parallel \\ G\times \mathcal{S} & & V_z \subset \mathcal{S} \end{array}$$




$\boxed{/\!/\!/}$

**Cor. 5.2**

(1) $\chi_{\mathcal{S}} : \mathcal{S} \to \mathfrak{h}/W$ is flat morphism.

(2) $S := \chi_{\mathcal{S}}^{-1}(0)$ has isolated sing. at $x \in \mathcal{O}_{\mathrm{sreg}}$.

**Proof.**

(1) • $\nu : G\times\mathcal{S} \to \mathfrak{g}$ is smooth.
$(g, s) \longmapsto \mathrm{Ad}(g)s$

$\because$ $d\nu : T_eG \oplus T_x\mathcal{S} \to T_x\mathfrak{g}$ is surj.
$(e, x) \in G\times\mathcal{S}$

In fact.

- $d\nu(T_eG \oplus 0) = T_x\mathcal{O}_x$
- $d\nu(0 \oplus T_x\mathcal{S}) = T_x\mathcal{S}$
- $T_x\mathfrak{g} = T_x\mathcal{O}_x \oplus T_x\mathcal{S}$ $\lrcorner$

For $z \in \mathcal{S}$, we take $z \in V \underset{\text{op.}}{\subseteq} \mathcal{S}$.

Then $\nu$ is smooth at all pts. of $\{e\} \times V$

For $z' \in \mathcal{S}$, $z' = \mu(t)z$ $(t \in \mathbb{C}^*)$

$\forall g \in G$, $(g, z') \in G \times \mathcal{S}$

$(g, z') \in \{G \times \mathbb{C}^* - \text{orbit of } (e, z)\}$

$\rightsquigarrow$ $\nu$ is smooth

$\rightsquigarrow$ $\nu$ is flat morphism

$$\begin{tikzcd}
G \times \mathcal{S} \arrow[r, "\nu"] \arrow[dr, "p_2"'] & \mathfrak{g} \arrow[r, "\chi"] & \mathfrak{h}/W \\
& \mathcal{S} \arrow[u, phantom, "\cup" description] \arrow[ur, "\chi_{\mathcal{S}}"'] &
\end{tikzcd}$$

(proj. to $\mathcal{S}$)

$\chi \circ \nu$ : flat $\because$ $\chi, \nu$ : flat

$\chi_{\mathcal{S}}$ : flat $\because$ $p_2$ : flat.

(2) • $\mathcal{O}_{reg} = \{ x \in \mathcal{N} \mid x : \text{nonsingular} \}$

• $\mathrm{Sing}(S) = \{ x \in S \mid x : \text{singular pt.} \}$

$\dim(\mathrm{Sing}(S)) = 0$

$\because$ • $S = \mathcal{S} \cap \mathcal{N}$ $(\because \mathcal{N} = \chi^{-1}(0))$

• $\dim \mathcal{O}_x + \dim \mathcal{S} \geq \dim \mathfrak{g}$ $(x \in S = \mathcal{S} \cap \mathcal{N})$

$\because$ $T_x \mathcal{O}_x + T_x \mathcal{S} = T_x \mathfrak{g}$ by Prop 5.1 $\overset{\text{IIS}}{\mathfrak{g}}$

$\dim \mathcal{S} = \ell + 2$

$\rightsquigarrow \quad \dim \mathcal{O}_x + (\ell+2) \geq \dim \mathfrak{g}$

$\dim \mathcal{O}_x \geq (\dim \mathfrak{g} - \ell) - 2$

$\dim \mathcal{O}_x \geq \dim \mathcal{N} - 2$

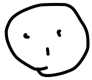
$\rightsquigarrow \quad \operatorname{codim} \mathcal{O}_x \leq 2.$

$\rightsquigarrow \quad x \in \mathcal{O}_{reg}$ or $x \in \mathcal{O}_{sreg}$

- $\dim \mathcal{S} = 2$
- $\operatorname{codim}(\operatorname{Sing}(\mathcal{N})) \geq 2$

$\rightsquigarrow \quad \operatorname{Sing}(\mathcal{S}) = \{x\}$

$\mathcal{S}$ has isolated sing.



$$\chi_S : S \to \mathfrak{h}/W$$

$$z \mapsto (\bar{\chi}_1(z), \dots, \bar{\chi}_\ell(z))$$

$$\bar{\chi}_i := \chi_i|_S$$

- weight of $\mathbb{C}^*$-action on $\chi_S$ is

  $(2d_1, \dots, 2d_\ell \,;\, n_1+1, \dots, n_{\ell+2}+1)$

- $2d_\ell > n_i + 1 \qquad (i = 1, \dots, \ell+2)$

- $s_1, \dots, s_{\ell+2}$ : a coordinate of $S$

  ($\dim S = \ell+2$)

$$\frac{\partial \bar{\chi}_\ell}{\partial s_i}(0) = 0 \qquad i = 1, \dots, \ell+2$$

$\rightsquigarrow$ $S$ has singular pt

$$\mathbb{C}^* \overset{\mu}{\curvearrowright} \mathcal{S} \qquad s = (s_1, \dots, s_{\ell+2})$$

$$\mu(t)(s_1, \dots, s_{\ell+2}) = (t^{n_1+1} s_1, \dots, t^{n_{\ell+2}+1} s_{\ell+2})$$

$$(\mu(t)\overline{\chi}_i)(s) = t^{2d_i} \chi_i(s)$$

$$\frac{\partial(\mu(t)\overline{\chi}_\ell)}{\partial(\mu(t) s_i)} = \frac{\partial \overline{\chi}_\ell}{\partial s_i} \times \frac{t^{2d_\ell}}{t^{n_i+1}}$$

$$= \frac{\partial \overline{\chi}_\ell}{\partial s_i} t^{\underline{2d_\ell - (n_i+1)} > 0}$$

$$\lim_{t \to 0} \frac{\partial(\mu(t)\overline{\chi}_\ell)}{\partial(\mu(t) s_i)} = 0$$

$$\|$$

$$\frac{\partial \overline{\chi}_\ell}{\partial s_i}(0)$$

$\blacksquare$